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ALGORITHM ENGINEERING

Solving 2-Stage Stochastic Steiner Tree Problems by 2-Stage Branch&Cut

Bernd Zey *TU Dortmund*

joint work with

I. Bomze *University of Vienna*
M. Chimani *Friedrich-Schiller-University of Jena*
M. Jünger *University of Cologne*
I. Ljubić *TU Vienna*
P. Mutzel *TU Dortmund*

15th Combinatorial Optimization Workshop (Aussois), 2011

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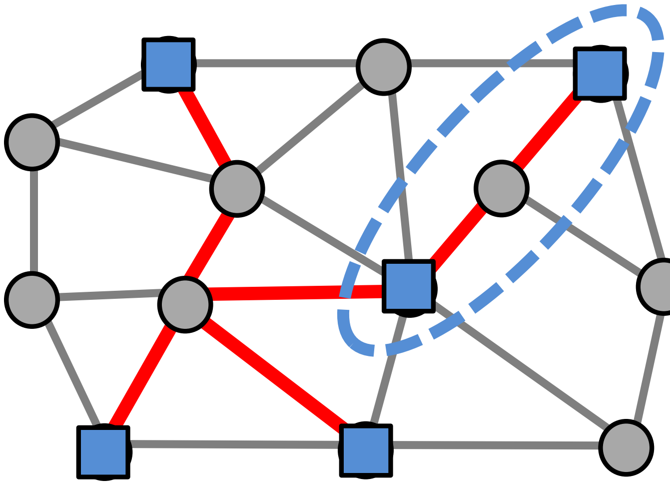
Deterministic Steiner Tree Problem (STP)

Given: undirected Graph G=(V,E)
positive edge costs c_e
set of terminals $R \subseteq V$, $R \neq \varnothing$

Objective: min {c(E_0) | $E_0 \subseteq E$, E_0 spans R}



Cut-based ILP formulation:

$$\min \sum_{e \in E} c_e x_e$$

$$\sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \subseteq V, \emptyset \neq S \cap R \neq R$$

$$x_e \in \{0,1\} \quad \forall e \in E$$

Deterministic Steiner Tree Problem (STP)

Given: undirected Graph G=(V,E)
positive edge costs c_e
set of terminals $R \subseteq V$, $R \neq \emptyset$

Objective: min $\{c(E_0) \mid E_0 \subseteq E, E_0$ spans $R\}$

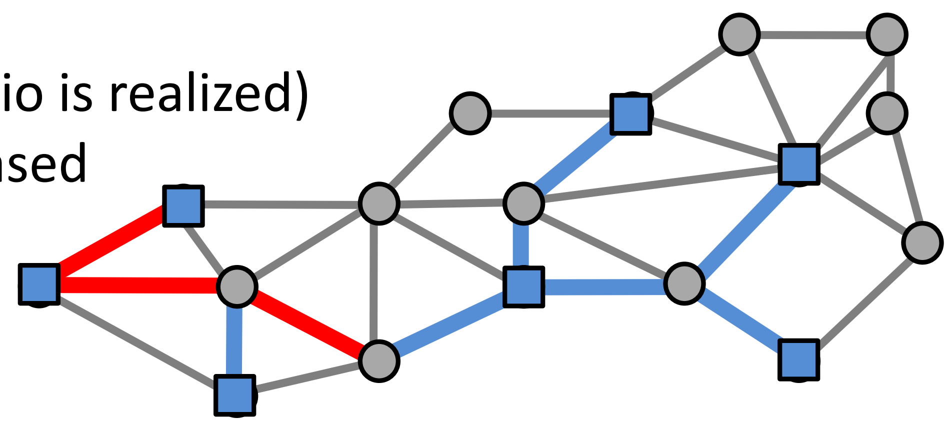
- (Decision problem) NP-complete
- Approximation Algorithms, ln(4)+ε≈1,39 [Byrka,Grandoni,Rothvoß,Sanità,2010]
- Fast exact algorithms based on e.g. [Polzin,Daneshmand,2001]
  - Branch-and-Cut, Dual Ascent [Uchoa,Aragão,Ribeiro,2002]
- Many applications for STP
  - Telecommunications, power supply, ...
  - ”Given the locations of a *root* (*server*, *powerplant*, *depot*,…) and the *customers*, connect the customers to it“

Stochastic Steiner Tree Problem (SSTP)

- In practice:
  - No precise knowledge of future customer demands
  - Future edge costs more expensive and volatile
- One possible approach: **Stochastic Optimization**
  - Estimate possible outcomes:
    Derive scenarios $\rightarrow$ terminals, edge costs

$\Rightarrow$ **2-Stage Stochastic Program**:

- **1st stage** (“now“)
  $\rightarrow$ buy cheap/profitable edges now
  $\rightarrow$ difficulty*: uncertainty*
- **2nd stage** (“future“, one scenario is realized)
  $\rightarrow$ additional edges are purchased
  (*recourse*)



Stochastic Steiner Tree Problem (SSTP)

Given: undirected Graph $G=(V,E)$, root $r \in V$, positive edge costs c_e, set of k scenarios $\mathcal{K}$:

probability p_k, edge costs $q_e^k \geq c_e$, set of terminals $R_k \subseteq V$, $r \in R_k$

Objective: find E_0 such that *expected costs* are minimized

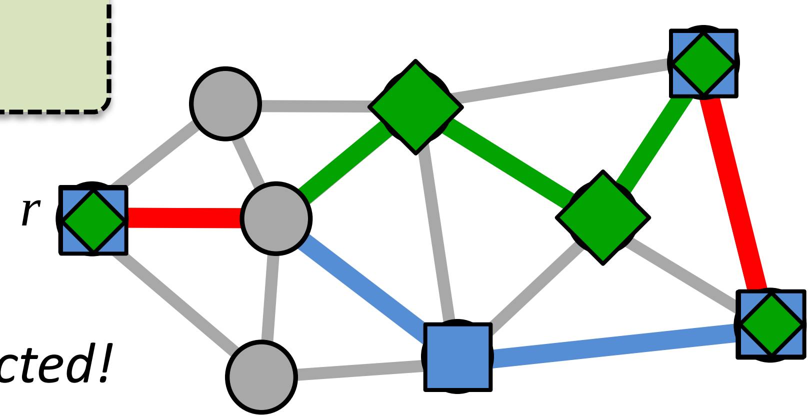
E_0: edges purchased in 1st stage

E_k: edges purchased in 2nd stage (scenario k)

$$\min \sum_{e \in E_0} c_e + \sum_{k \in \mathcal{K}} p_k \sum_{e \in E_k} q_e^k$$

$$\text{s.t. } (E_0 \cup E_k) \text{ spans } R_k$$

$\Rightarrow$ *1st stage might be disconnected!*



Stochastic Steiner Tree Problem (SSTP)

Given: undirected Graph $G=(V,E)$, root $r \in V$, positive edge costs c_e, set of k scenarios $\mathcal{K}$:

probability p_k, edge costs $q_e^k \geq c_e$, set of terminals $R_k \subseteq V$, $r \in R_k$

Objective: find E_0 such that *expected costs* are minimized

- NP-complete ($\rightarrow$ STP)
- In general: hard to approximate

  $\rightarrow$ Label Cover: $\Omega(2^{\log^{1-\epsilon} n})$ [Gupta,Hajiaghayi,Kumar,2007]
- Constant factor approximation for special cases, e.g.
  - *fixed inflation factor*, i.e. $\exists \sigma_k$ s.t. $q_e^k = c_e \cdot \sigma_k$ [Gupta,Ravi,Sinha,2007]
    Primal-Dual Scheme on the undirected ILP formulation
  - 4-approximation algorithm [Shmoys,Swamy,2006]

here: general case with $q_e^k \geq c_e \ \forall\ k, e$

Our Contribution

- New, stronger ILP-formulation for SSTP
  - Semi-directed formulation
  - Undirected formulation [Gupta,Ravi,Sinha,2007]
- Exact algorithm for SSTP
  - (Benders-like) Decomposition
  - Concept of 2-Stage Branch&Cut
    - Applicable to many network design problems
- First computational study for SSTP
  - Exact solutions for instances with up to 165 nodes, 274 edges and 500 scenarios
  - Benchmark library

SSTP - Undirected Formulation

(Straight-Forward) **Undirected Model, Extended Form (UEF)**:

[Gupta,Ravi,Sinha,2007]

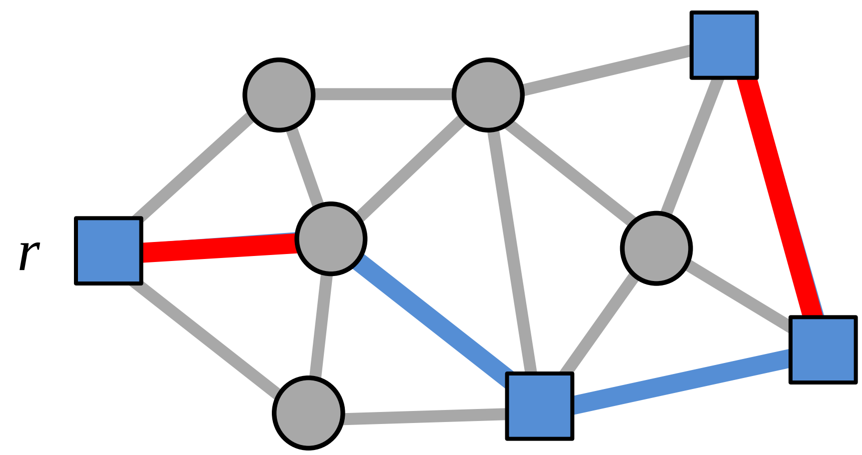
$$\min \sum_{e \in E} c_e x_e + \sum_{k \in \mathcal{K}} p_k \sum_{e \in E} q_e^k y_e^k$$

$$\sum_{e \in \delta(S)} (x_e + y_e^k) \geq 1 \qquad \forall k \in \mathcal{K}, \forall \emptyset \neq S \subset V \setminus \{r\}, S \cap R_k \neq \emptyset$$

$$x_e, y_e^k \in \{0,1\} \qquad \forall e \in E, \forall k \in \mathcal{K}$$

2nd stage variables $\rightarrow E_k$

1st stage variables $\rightarrow E_0$



Deterministic STP - Directed Formulations

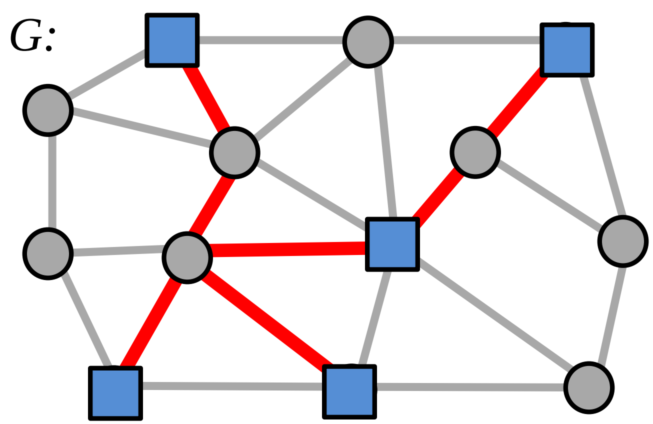
For STP and variants it is known:

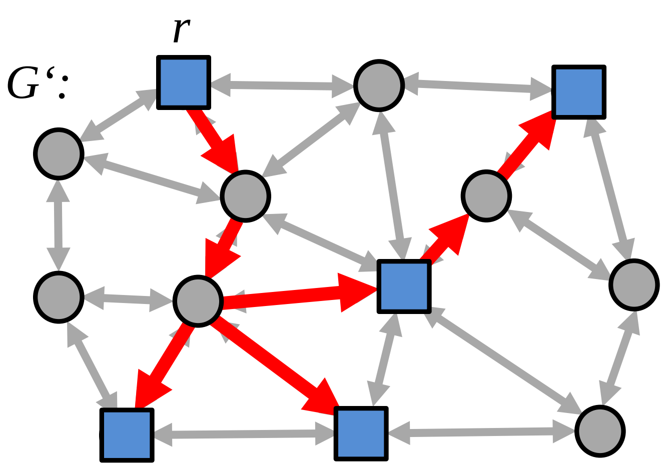
Directed formulations are stronger than undirected ones.

Undirected graph G=(V,E) $\Rightarrow$ Bidirected version of G: G'=(V,A).

$\Rightarrow$ ***Steiner Arborescence***:

A „Steiner tree" directed from the root outwards





$$\min \sum_{e \in E} c_e x_e$$

$$\sum_{e \in \delta(S)} x_e \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \cap R \neq \emptyset$$

$$x_e \in \{0,1\} \quad \forall e \in E$$

$$\min \sum_{e=\{u,v\} \in E} c_e(\bar{x}_{uv} + \bar{x}_{vu})$$

$$\sum_{a \in \delta^-(S)} \bar{x}_a \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \cap R \neq \emptyset$$

$$\bar{x}_a \in \{0,1\} \quad \forall a \in A$$

(S)STP - Directed Formulations

For STP and variants it is known:

Directed formulations are stronger than undirected ones.

Undirected graph G=(V,E) $\Rightarrow$ Bidirected version of G: G‘=(V,A).

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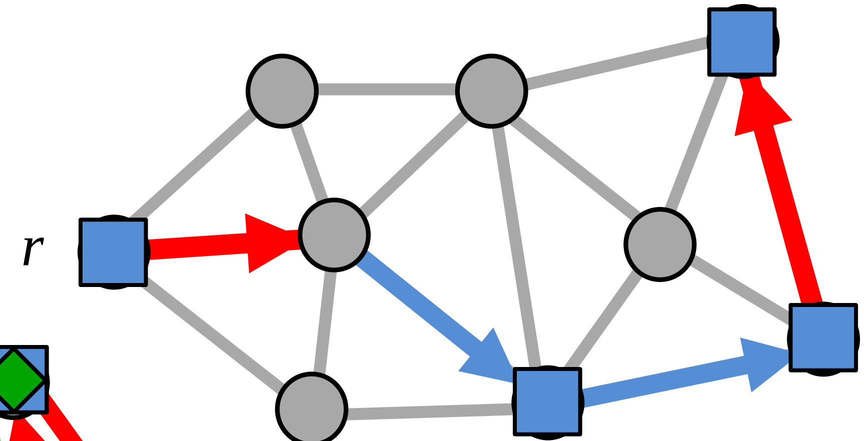
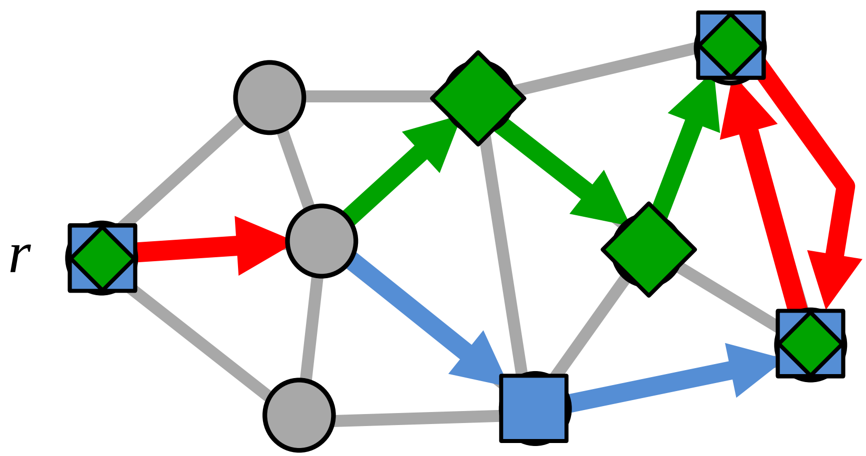
Applicable for the SSTP?

directed 2nd stage ✓

directed 1st stage !





SSTP - Semi-Directed Formulation

Semi-directed Model, Extended Form (SEF):

$$\min \sum_{e \in E} c_e x_e + \sum_{k \in \mathcal{K}} p_k \sum_{e=\{u,v\} \in E} q_e^k (z_{uv}^k + z_{vu}^k - x_e)$$

$$z^k(\delta^-(S)) \geq 1 \qquad \forall S \subseteq V \setminus \{r\}, S \cap R \neq \emptyset, \forall k \in \mathcal{K}$$ directed cuts

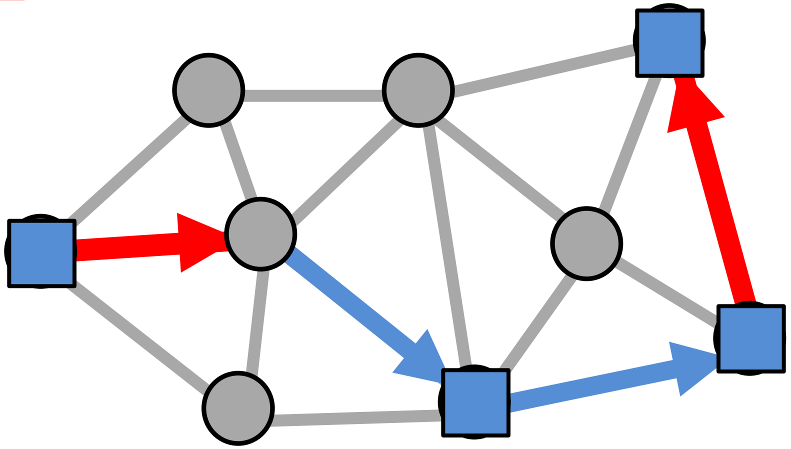
$$z_{uv}^k + z_{vu}^k \geq x_e \qquad \forall e = \{u,v\} \in E, \forall k \in \mathcal{K}$$ capacity constraints

$$z_{uv}^k \in \{0,1\} \qquad \forall (uv) \in A, \forall k \in \mathcal{K}$$

$$x_e \in \{0,1\} \qquad \forall e \in E$$

z_{uv}: 2nd stage variables (*directed*) $\rightarrow E_0 \cup E_k$

x_e: 1st stage variables (*undirected*) $\rightarrow E_0$



SSTP - Semi-Directed Formulation

Semi-directed Model, Extended Form (SEF):

$$\min \sum_{e \in E} c_e x_e + \sum_{k \in \mathcal{K}} p_k \sum_{e=\{u,v\} \in E} q_e^k (z_{uv}^k + z_{vu}^k - x_e)$$

$$z^k(\delta^-(S)) \geq 1 \quad \forall S \subseteq V \setminus \{r\}, S \cap R \neq \emptyset, \forall k \in \mathcal{K}$$

directed cuts

$$z_{uv}^k + z_{vu}^k \geq x_e \quad \forall e = \{u,v\} \in E, \forall k \in \mathcal{K}$$

capacity constraints

$$z_{uv}^k \in \{0,1\} \quad \forall (uv) \in A, \forall k \in \mathcal{K}$$

$$x_e \in \{0,1\} \quad \forall e \in E$$

z_{uv}: 2nd stage variables (*directed*) $\rightarrow E_0 \cup E_k$

x_e: 1st stage variables (*undirected*) $\rightarrow E_0$

Theorem

The semi-directed model is stronger than the undirected model, i.e.:

- $\forall$ instances: $LP_{relax}(SEF) \geq LP_{relax}(UEF)$,
- $\exists$ instances: $LP_{relax}(SEF) > LP_{relax}(UEF)$

Decomposition

We could solve (**SEF**) directly via Branch&Cut…

but: The ILP grows with the number of scenarios → impractical

⇒ **Decompose** the ILP (similar to Benders Decomposition)

$$\min \sum_{e \in E} c_e x_e + \sum_{k \in \mathcal{K}} p_k \sum_{e=\{u,v\} \in E} q_e^k (z_{uv}^k + z_{vu}^k - x_e)$$

$$z^k(\delta^-(S)) \geq 1 \qquad \forall S \subseteq V \setminus \{r\}, S \cap R \neq \emptyset, \forall k \in \mathcal{K}$$

$$z_{uv}^k + z_{vu}^k \geq x_e \qquad \forall e = \{u,v\} \in E, \forall k \in \mathcal{K}$$

$$z_{uv}^k \in \{0,1\} \qquad \forall (uv) \in A, \forall k \in \mathcal{K}$$

$$x_e \in \{0,1\} \qquad \forall e \in E$$

1st stage

2nd stage

restricted Steiner Arborescence Problem

Master Problem

$$\min \sum_{e \in E} c_e x_e + \sum_{k \in \mathcal{K}} p_k \theta_k$$

$$f_k(x) \leq \theta_k \qquad \forall k \in \mathcal{K}, \ \text{some } f_k(\cdot)$$

$$x_e \in \{0,1\} \qquad \forall e \in E$$

$$\theta_k \geq 0 \qquad \forall k \in \mathcal{K}$$

Subproblem, scenario k, fixed 1st stage sol. x'

$$Q_k(x') = \min \sum_{e=\{u,v\} \in E} q_e^k (z_{uv}^k + z_{vu}^k - x'_e)$$

$$z^k(\delta^-(S)) \geq 1 \qquad \forall S \subseteq V \setminus \{r\}, S \cap R_k \neq \emptyset, \forall k \in \mathcal{K}$$

$$z_{uv}^k + z_{vu}^k \geq x'_e \qquad \forall e = \{u,v\} \in E, \forall k \in \mathcal{K}$$

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Decomposition

Master Problem

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Subproblem, scenario k, fixed 1st stage sol. x‘

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$$z_{uv}^k + z_{vu}^k \geq x'_e \quad \forall e = \{u,v\} \in E, \forall k \in \mathcal{K}$$

$$z_{uv}^k \in \{0,1\} \quad \forall (uv) \in A, \forall k \in \mathcal{K}$$

What kind of function $f_k(\cdot)$ do we need?

Integer Optimality cut: [Laporte,Louveaux,1993]

Cuts off non-optimum (integer) solution (x‘,θ)

$$Q_k(x') \cdot id(x, x') \leq \theta_k$$

with $id(x,x')=1$ iff $x=x'$

L-shaped optimality cuts:

- *Purpose*: Improve lower bounds on 2nd-stage costs $\Rightarrow$ raise θ-variables
- Generated by solving the relaxed 2nd-stage subproblems (*restricted Steiner Arborescence Problems*) and deducing subgradients

Decomposition

Master Problem

$$\min \sum_{e\in E} c_e x_e + \sum_{k\in\mathcal{K}} p_k \theta_k$$

$$f_k(x) \le \theta_k \quad \forall k \in \mathcal{K},\ \text{some } f_k(\cdot)$$

$$x_e \in \{0,1\} \quad \forall e \in E$$

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Subproblem, scenario k, fixed 1st stage sol. x'

$$Q_k(x') = \min \sum_{e=\{u,v\}\in E} q_e^k (z_{uv}^k + z_{vu}^k - x_e')$$

$$z^k(\delta^-(S)) \ge 1 \quad \forall S \subseteq V \setminus \{r\}, S \cap R_k \ne \emptyset, \forall k \in \mathcal{K}$$

$$z_{uv}^k + z_{vu}^k \ge x_e' \quad \forall e = \{u,v\} \in E, \forall k \in \mathcal{K}$$

$$z_{uv}^k \in \{0,1\} \quad \forall (uv) \in A, \forall k \in \mathcal{K}$$

What kind of function $f_k(\cdot)$ do we need?

L-shaped optimality cuts:

→ Solve Master-LP (→ solution x')

→ solve all $|\mathcal{K}|$ LP-relaxations of the 2nd-Stage-ILPs:
objective function $R_k(x')$, dual variable values α^k_S, β^k_e

→ generate cuts if $R_k(x') > \theta_k$

Lemma: $\beta^k - q^k \in \partial R_k(x')$, i.e., it is a subgradient at x'

$(\Sigma_{k\in\mathcal{K}} p_k(\beta^k - q^k) \in \partial R(x'))$

$$\sum_{S\subseteq V\setminus\{r\}: S\cap R\ne\emptyset} \alpha_S^k + \sum_{e\in E} (\beta_e^k - q_e^k) x_e \le \theta_k$$

2-Stage Branch&Cut

Master Problem

$$\min \sum_{e\in E} c_e x_e + \sum_{k\in\mathcal{K}} p_k \theta_k$$

$$f_k(x) \leq \theta_k \quad \forall k \in \mathcal{K}, \text{ some } f_k(\cdot)$$

$$x_e \in \{0,1\} \quad \forall e \in E$$

$$\theta_k \geq 0 \quad \forall k \in \mathcal{K}$$

Subproblem, scenario k, fixed 1st stage sol. x'

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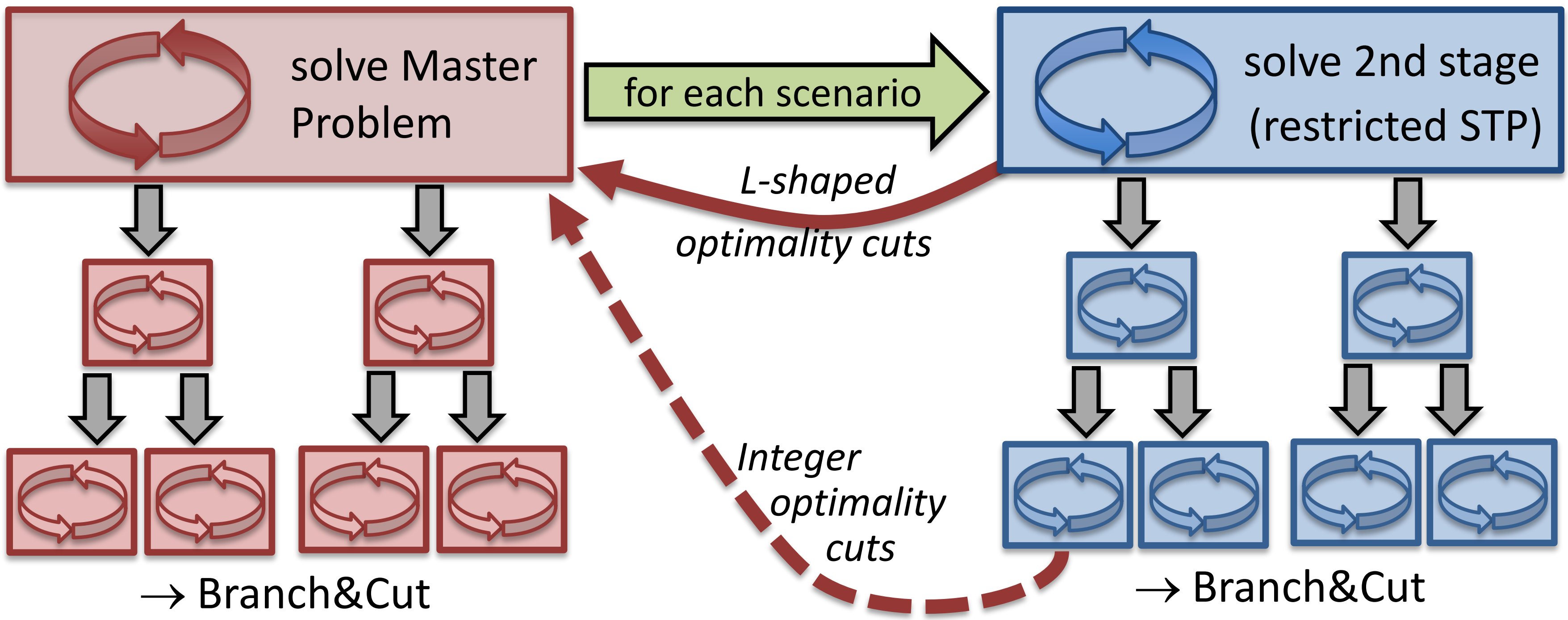


Alternative Decomposition

Alternative Decomposition

q_e* are the expected 2nd-stage costs, i.e., $q_e^* := \sum_{k \in \mathcal{K}} p_k \cdot q_e^k$

2BC $$\min \sum_{e \in E} c_e x_e + \sum_{k \in \mathcal{K}} p_k \sum_{e=\{u,v\} \in E} q_e^k (z_{uv}^k + z_{vu}^k - x_e)$$

1st-stage Master — 2nd-stage Subproblems

2BC* $$\min \sum_{e \in E} (c_e - q_e^*) x_e + \sum_{k \in \mathcal{K}} p_k \sum_{e=\{u,v\} \in E} q_e^k (z_{uv}^k + z_{vu}^k)$$

Some properties of **2BC***

- Counter-intuitive 1st-stage: negative coefficients in the objective function $\rightarrow$ *maximize* number of edges
- L-shaped cuts are sparser $\rightarrow$ beneficial in practice

Experiments – SSTPLib

Machine/Implementation

- Intel Core-i7 2.67GHz Quad Core, 12 GB RAM, Ubuntu 9.04
- 32bit (2GB RAM), single core/job
- C++, generic B&C framework: ABACUS 3.0, LP solver: IBM CPLEX 10.1

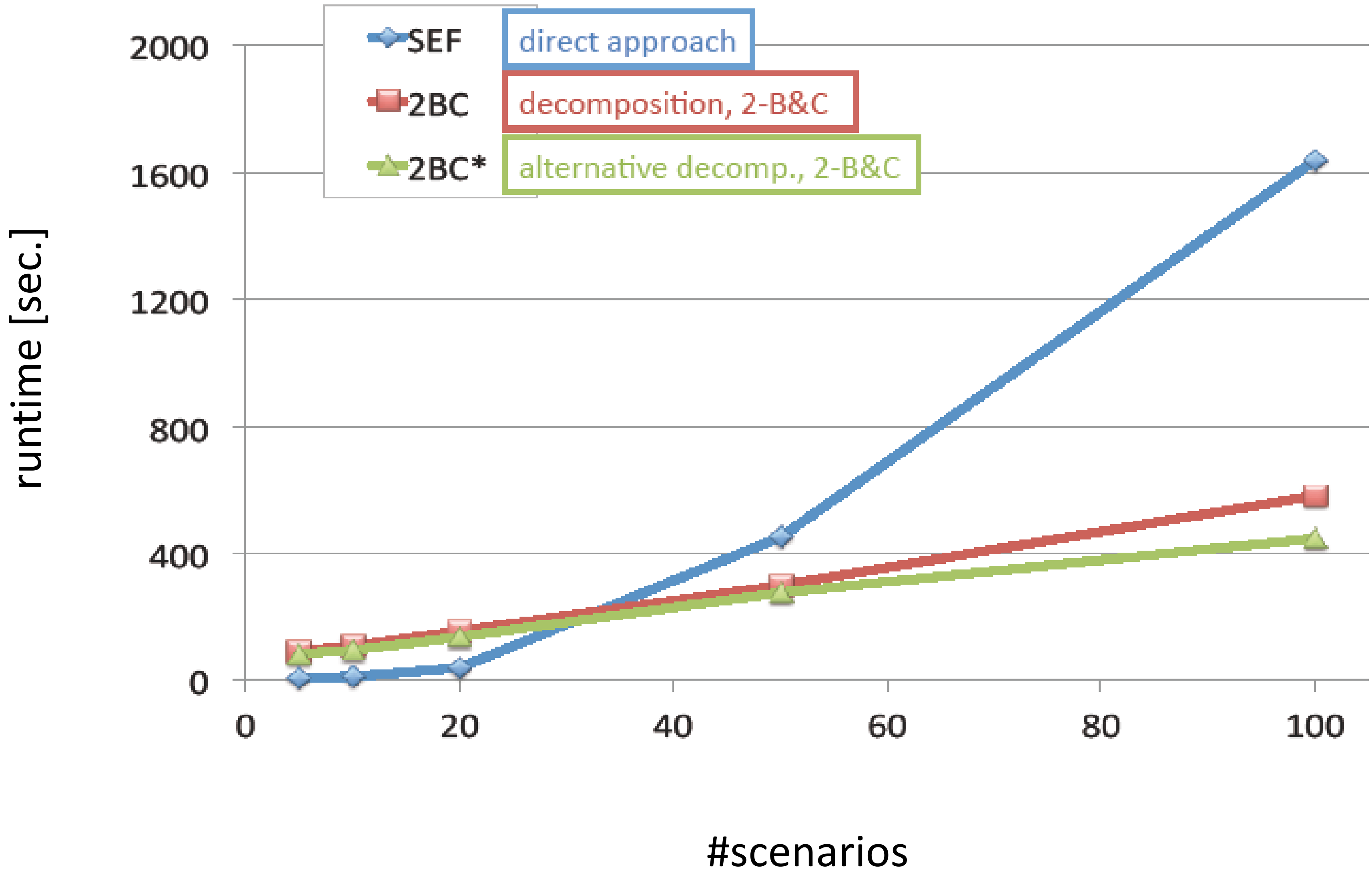
Deterministic Instances:

- Groups K, P (Prize-Collecting STP, preprocessed): 91 nodes, 237 edges
- lin instances (SteinLib): 165 nodes, 274 edges

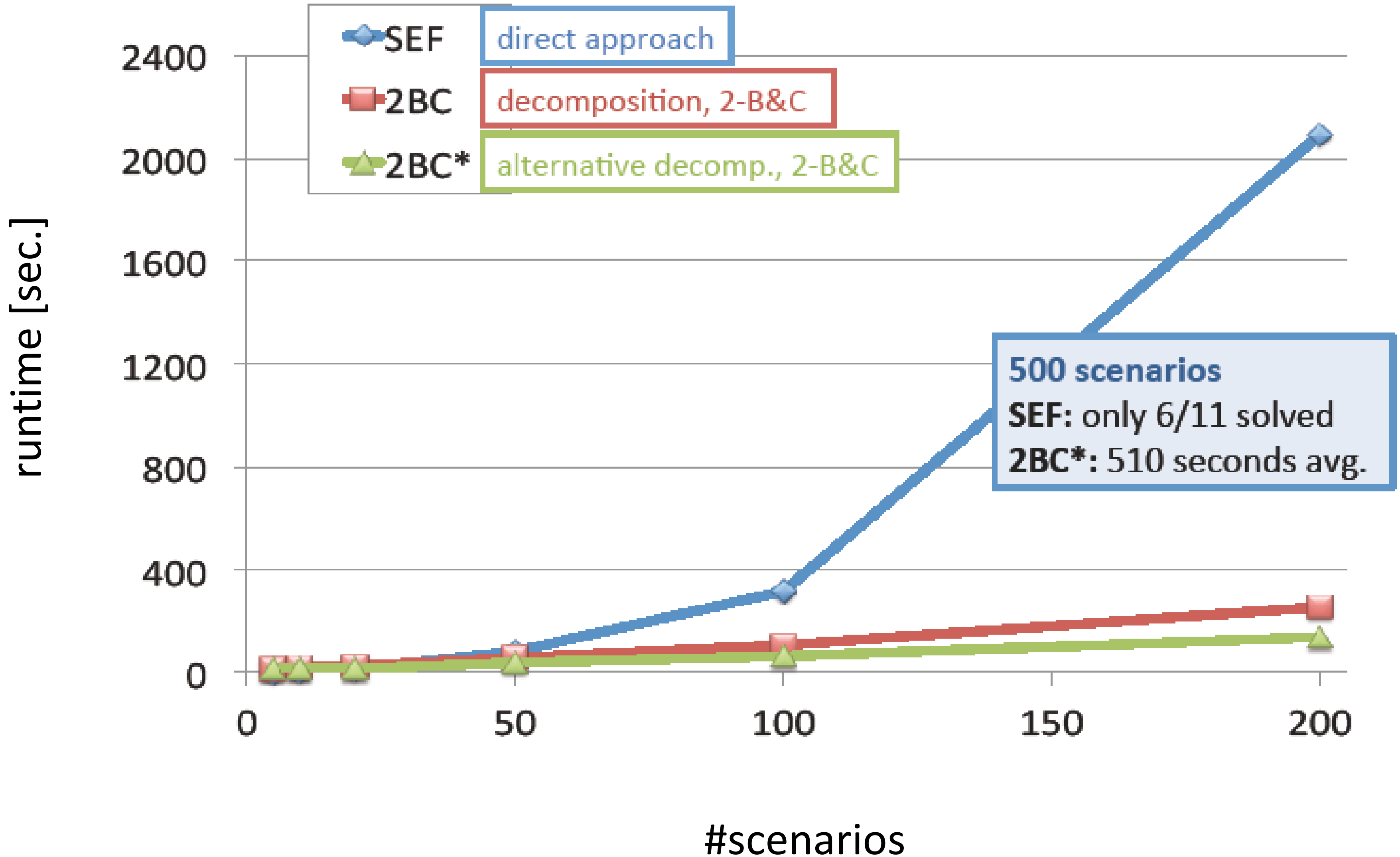
Deterministic ⇝ Stochastic:

- Generate k scenarios
- Obtain p_k by distributing 1000 points randomly (uniform)
- Random set of terminals: terminal/steiner nodes 30%/5%
- Edge costs q_e^k from $[1.1c_e, 1.3c_e]$

Experiments – P group (average)



Experiments – K group (average)



Conclusion

Theory

- New, provably stronger formulations (2BC, 2BC*)
- New solution framework 2-Stage Branch-and-Cut

Experiments

- First computational study; benchmark library
- On tested instances:
  - Decomposition beneficial for instances with >20 scenarios
  - 2BC* decomposition dominates 2BC

Challenges

- SSTP: Stronger, purely directed formulations?
- SSTP: Extensions for tighter 2nd stage formulations?
- SSTP: Better approximation with semi-directed formulation?
- 2-Stage B&C: **Not only SSTP**! Applicable to all Goemans-Williamson-type network design problems (Prize-Collecting STP/TSP, Survivable Network Design,...)

Thank you for your attention