Set covering with ordered replacement:

Additive and Multiplicative gaps

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Joint work with:

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Aussois, 2011

General Set covering problem

Input:

- A ground set of items $\{1, \ldots, n\}$;
- A set system $\mathcal{S} = \{S_1, \ldots, S_m\}$, with $S_j \subseteq \{1, \ldots, n\}$;

Goal:

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• Approximation: $O(\log n)$[Chvátal, '79], corresponding hardness result [Feige '98]

Linear Programming Relaxation

- Let $\chi(S) \in \{0,1\}^n$ be the characteristic vector of a set $S \in \mathcal{S}$

$$OPT_f(\mathcal{S}) \quad = \quad \min\Big\{ \sum_{S \in \mathcal{S}} x_S \mid \sum_{S \in \mathcal{S}} x_S \cdot \chi(S) \geq \mathbf{1}, x \geq \mathbf{0} \Big\}$$

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...but there are some set covering problems more tractable!

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Goal: Pack items into minimum number of **bins** of size 1.

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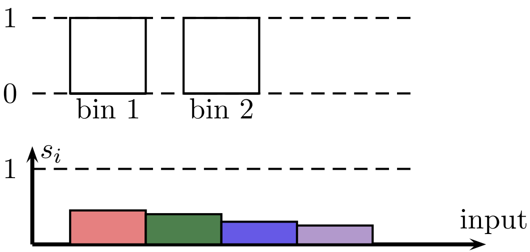

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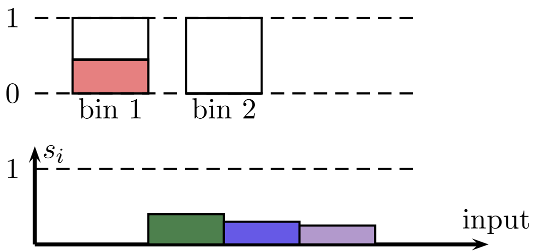

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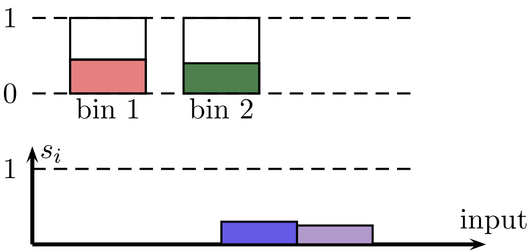

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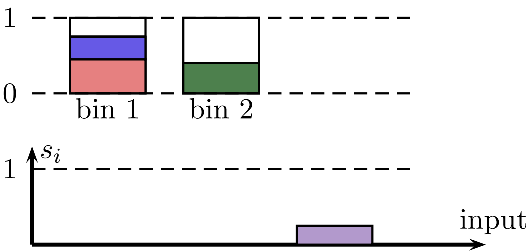

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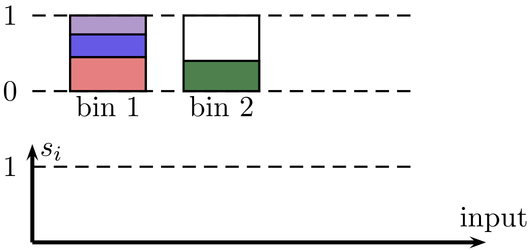

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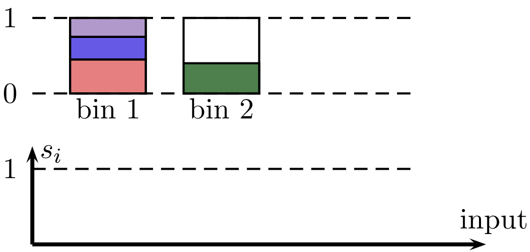

- Approximation: Asymptotic FPTAS [Karmarkar & Karp '82]:
$APX \leq OPT + O(\log^2 n)$ in poly-time

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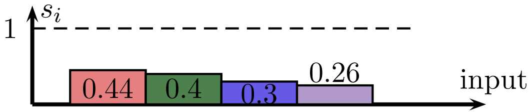

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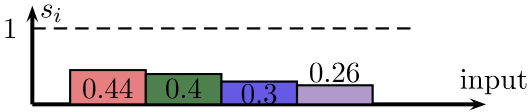

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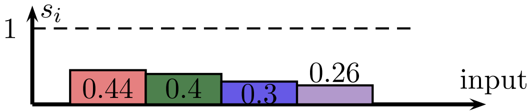

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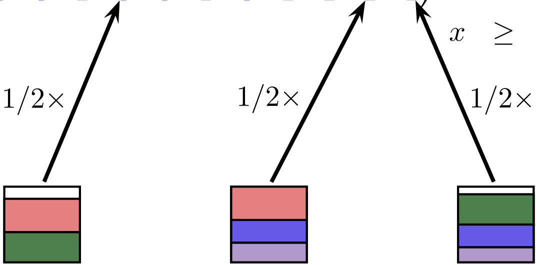

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...Possible answer: the **ordered replacement**!

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What can we say in this general setting?

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For the Set cover with ordered replacement problem, the multiplicative integrality gap is $\Theta(\log \log n)$.

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- As in [Karmarkar & Karp '82], we will construct an integer solution from a fractional one by doing a sequence of iterations.
- At each iteration:
  - cover part of our elements by rounding down a fractional solution;
  - modify the residual instance;
  - re-optimize.

Preliminaries of the proof

- We will consider the following more general LP:

$$\min \sum_{S \in \mathcal{S}} x_S$$

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- ...but the grouping of [K & K '82] crucially relies on the given item sizes of the Bin packing instace, which are missing here...!!

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Lemma

Let $\mathcal{S}$ be an instance of Set cover with ordered replacement. Let s be a vector of pseudo-sizes satisfying (i) and (ii), and $\alpha := \max_{S \in \mathcal{S}} \sum_{i \in S} s_i$. Then $OPT(\mathcal{S}, b) - OPT_f(\mathcal{S}, b) = O(\alpha \cdot \log(\frac{1}{s_{min}}) \cdot \log n)$.

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- Total size of discarded elements $\leq 8 \cdot \alpha \cdot (\log(1/s_{\min}) + 1)$. By *(ii)*, we can cover them with $O(\alpha \cdot \log(1/s_{\min}))$ sets.

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(discarded groups compensate the round-up operation!)

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- By the Lemma,

$$\text{additive gap is } O(\alpha \cdot \log(\tfrac{1}{s_{min}}) \cdot \log n) = O(\log^3 n).$$

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- Open questions:
  - Is there a $O(1)$-apx for Set cover with ordered replacement? (we can prove a 2-apx in quasi-polynomial time)
  - Can we give a bound on the additive integrality gap better than $O(\log^3 n)$?
  - Is the additive integrality gap constant for Bin packing?