Reverse multistar inequalities and Vehicle Routing Problem with lower bound capacities

Gouveia, Riera, Salazar
(University of Lisbon, Portugal; University of La Laguna, Spain)

Aussois, January 10-14, 2011

Outline

Definition

$G = (V, A)$ directed graph
$V = \{1, 2, \ldots, n\}$ with 1 the depot and $2, \ldots, n$ customers.
Each node $i \in V$ has a demand d_i : $\sum_{i \in V} d_i = 0$.
Each arc $a \in A$ has a cost c_a.
All vehicles are identical, with only an upper vehicle capacity $\overline{Q}$.
Character m denotes the number of required vehicles. This value may be a-priori fixed, or an unknown value.

Definition

$G = (V, A)$ directed graph
$V = \{1, 2, \ldots, n\}$ with 1 the depot and $2, \ldots, n$ customers.
Each node $i \in V$ has a demand d_i : $\sum_{i \in V} d_i = 0$.
Each arc $a \in A$ has a cost c_a.
All vehicles are identical, with only an upper vehicle capacity $\overline{Q}$.
Character m denotes the number of required vehicles. This value may be a-priori fixed, or an unknown value.

The Classical VRP looks for a set of routes, all passing through 1, such that each customer is in exactly one route, the sum of demands for customers in each route is not bigger than $\overline{Q}$, and the sum of the costs of arcs in routes is minimized.

SCF formulation (Gavish & Graves, 1979)

$$\min \sum_{a \in A} c_a x_a$$

$$x(1 : V \setminus \{1\}) = x(V \setminus \{1\} : 1) = m$$

$$x(i : V \setminus \{i\}) = x(V \setminus \{i\} : i) = 1 \qquad \text{for all } i \in V \setminus \{1\}$$

$$x_a \in \{0, 1\} \qquad \text{for all } a \in A$$

$$\sum_{a \in \delta^-(i)} f_a - \sum_{a \in \delta^+(i)} f_a = d_i \qquad \text{for all } i \in V$$

$$\underline{q}_a x_a \leq f_a \leq \overline{q}_a x_a \qquad \text{for all } a \in A$$

where $\underline{q}_a$ and $\overline{q}_a$ are appropriated values (functions of $\overline{Q}$).

Projecting the SCF variables

Theorem (Hoffman 1960)

There is a solution $f = [f_a : a \in A]$ *of the linear system*

$$\sum_{a \in \delta^-(i)} f_a - \sum_{a \in \delta^+(i)} f_a = d_i \qquad \text{for all } i \in V$$

$$\underline{q}_a x_a \leq f_a \leq \overline{q}_a x_a \qquad \text{for all } a \in A$$

if and only if

$$\sum_{a \in \delta^-(S)} \overline{q}_a x_a \geq \sum_{a \in \delta^+(S)} \underline{q}_a x_a + \sum_{i \in S} d_i \qquad \text{for all } S \subset V. \tag{1}$$

Projecting the SCF variables

Theorem (Hoffman 1960)

There is a solution $f = [f_a : a \in A]$ of the linear system

$$\sum_{a \in \delta^-(i)} f_a - \sum_{a \in \delta^+(i)} f_a = d_i \qquad \text{for all } i \in V$$

$$\underline{q}_a x_a \leq f_a \leq \overline{q}_a x_a \qquad \text{for all } a \in A$$

if and only if

$$\sum_{a \in \delta^-(S)} \overline{q}_a x_a \geq \sum_{a \in \delta^+(S)} \underline{q}_a x_a + \sum_{i \in S} d_i \qquad \text{for all } S \subset V. \qquad (1)$$

Equivalently: ... if and only if

$$\sum_{a \in \delta^-(S)} \underline{q}_a x_a \leq \sum_{a \in \delta^+(S)} \overline{q}_a x_a + \sum_{i \in S} d_i \qquad \text{for all } S \subset V. \qquad (2)$$

Observations

Inequality (2) associated with S coincides with inequality (1) associated with the set $V \setminus S$.

Observations

Inequality (2) associated with S coincides with inequality (1) associated with the set $V \setminus S$.

We can divide each family of inequalities in two groups:

- one group is associated with the sets S containing node 1
- the other group with the sets S not containing node 1.

Then a complete projection is given by (1) and (2) for all $S : 1 \notin S$.

Observations

Inequality (2) associated with S coincides with inequality (1) associated with the set $V \setminus S$.

We can divide each family of inequalities in two groups:

- one group is associated with the sets S containing node 1
- the other group with the sets S not containing node 1.

Then a complete projection is given by (1) and (2) for all $S : 1 \notin S$.

From now on, (1) with $1 \notin S$ will be called Multistar (MS) inequalities and (2) with $1 \notin S$ will be called Reverse Multistar (RMS) inequalities.

Observations

Inequality (2) associated with S coincides with inequality (1) associated with the set $V \setminus S$.

We can divide each family of inequalities in two groups:

- one group is associated with the sets S containing node 1
- the other group with the sets S not containing node 1.

Then a complete projection is given by (1) and (2) for all $S : 1 \notin S$.

From now on, (1) with $1 \notin S$ will be called Multistar (MS) inequalities and (2) with $1 \notin S$ will be called Reverse Multistar (RMS) inequalities.

- The MS inequalities (1) are useful in cutting-plane approaches (see, e.g., Letchford, Eglese and Lysgaard [2002]).
- However, the RMS inequalities (2) are useless (they are implied by other inequalities in the model).

MS and RMS inequalities

Using

$$d_i := \begin{cases} 1 & \text{if } i \in V \setminus \{1\} \\ 1 - |V| & \text{if } i = 1 \end{cases}$$

and

$$\begin{aligned} \underline{q}_a = \overline{q}_a = 0 \quad & \text{for all } a \in \delta^-(1) \\ \underline{q}_a = 1 \text{ and } \overline{q}_a = \overline{Q} \quad & \text{for all } a \in \delta^+(1) \\ \underline{q}_a = 1 \text{ and } \overline{q}_a = \overline{Q} - 1 \quad & \text{for all } a \notin \delta^+(1) \cup \delta^-(1). \end{aligned}$$

MS and RMS inequalities

Using

$$d_i := \begin{cases} 1 & \text{if } i \in V \setminus \{1\} \\ 1 - |V| & \text{if } i = 1 \end{cases}$$

and

$$\begin{aligned} \underline{q}_a = \overline{q}_a = 0 & \quad \text{for all } a \in \delta^-(1) \\ \underline{q}_a = 1 \text{ and } \overline{q}_a = \overline{Q} & \quad \text{for all } a \in \delta^+(1) \\ \underline{q}_a = 1 \text{ and } \overline{q}_a = \overline{Q} - 1 & \quad \text{for all } a \notin \delta^+(1) \cup \delta^-(1). \end{aligned}$$

The MS inequalities (1) are:

$$\overline{Q}x(1:S) + (\overline{Q} - 1)x(S':S) \geq x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}.$$

The RMS inequalities (2) are:

$$x(1:S) + x(S':S) \leq (\overline{Q} - 1)x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}.$$

Definition

The *Balanced Vehicle Routing Problem* (BVRP) is the classical VRP where each vehicle has also a lower capacity Q.

MS and RMS inequalities

Using

$$d_i := \begin{cases} 1 & \text{if } i \in V \setminus \{1\} \\ 1 - |V| & \text{if } i = 1 \end{cases}$$

and

$$\begin{aligned} \underline{q}_a = \overline{q}_a = 0 & \quad \text{for all } a \in \delta^-(1) \\ \underline{q}_a = \underline{Q} \text{ and } \overline{q}_a = \overline{Q} & \quad \text{for all } a \in \delta^+(1) \\ \underline{q}_a = 1 \text{ and } \overline{q}_a = \overline{Q} - 1 & \quad \text{for all } a \notin \delta^+(1) \cup \delta^-(1). \end{aligned}$$

MS and RMS inequalities

Using

$$d_i := \begin{cases} 1 & \text{if } i \in V \setminus \{1\} \\ 1 - |V| & \text{if } i = 1 \end{cases}$$

and

$$\begin{aligned} \underline{q}_a = \overline{q}_a = 0 & \quad \text{for all } a \in \delta^-(1) \\ \underline{q}_a = \underline{Q} \text{ and } \overline{q}_a = \overline{Q} & \quad \text{for all } a \in \delta^+(1) \\ \underline{q}_a = 1 \text{ and } \overline{q}_a = \overline{Q} - 1 & \quad \text{for all } a \notin \delta^+(1) \cup \delta^-(1). \end{aligned}$$

The MS inequalities (1) are:

$$\overline{Q}x(1:S) + (\overline{Q} - 1)x(S' : S) \geq x(S : S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}.$$

The RMS inequalities (2) are:

$$\underline{Q}x(1:S) + x(S' : S) \leq (\overline{Q} - 1)x(S : S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}.$$

Enhanced RMS inequalities

The RMS inequalities (2) are:

$$\underline{Q}x(1:S) + x(S':S) \leq (\overline{Q}-1)x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}.$$

Enhanced RMS inequalities

The RMS inequalities (2) are:

$$\underline{Q}x(1:S)+x(S':S)\leq(\overline{Q}-1)x(S:S')+|S| \qquad \text{for all } S\subseteq V\setminus\{1\}.$$

Then Enhanced RMS inequalities are:

$$\underline{Q}x(1:S)+x(S':S)\leq(\underline{Q}-1)x(S:S')+|S| \qquad \text{for all } S\subseteq V\setminus\{1\}. \tag{3}$$

Enhanced RMS inequalities

The RMS inequalities (2) are:

$$\underline{Q}x(1:S) + x(S':S) \leq (\overline{Q}-1)x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}.$$

Then Enhanced RMS inequalities are:

$$\underline{Q}x(1:S) + x(S':S) \leq (\underline{Q}-1)x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}. \tag{3}$$

Proposition

ERMS inequalities (3) are valid for BVRP

Enhanced RMS inequalities

The RMS inequalities (2) are:

$$Qx(1:S) + x(S':S) \leq (\overline{Q}-1)x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}.$$

Then Enhanced RMS inequalities are:

$$Qx(1:S) + x(S':S) \leq (Q-1)x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}. \tag{3}$$

Proposition

ERMS inequalities (3) are valid for BVRP

Open Question

separation problem of (3)

Special case: $\underline{Q} = \overline{Q}$

Proposition

When $\underline{Q} = \overline{Q}(= Q)$, the ERMS inequality (3) for set S is equivalent to the MS inequality (1) for the set S'.

Proof:
Using the degree constraint for the depot

$$x(1 : V \setminus \{1\}) = (|V| - 1)/Q$$

Then the ERMS inequality (3) for S is

$$|V| - 1 + x(S' : S) \leq (Q - 1)x(S : S') + Qx(1 : S') + |S|$$

Since $|V| - 1 - |S| = |S'|$, this ineq is the MS inequality (1) for S'.

General approach

- Take any valid inequality:

$$\sum_i \alpha_i x_i \leq \alpha_0$$

- Take a positive number β.
- Divide all α_i and β by β and round down all new numbers:

$$\sum_i \left\lfloor \frac{\alpha_i}{\beta} \right\rfloor x_i \leq \left\lfloor \frac{\alpha_0}{\beta} \right\rfloor$$

Example 1: known inequalities

Take a MS inequality and divide by $\overline{Q}$. The rounded inequality is:

$$x(1:S) + \left\lceil \frac{\overline{Q}-1}{\overline{Q}} \right\rceil x(S':S) \geq \left\lfloor \frac{1}{\overline{Q}} \right\rfloor x(S:S') + \left\lceil \frac{|S|}{\overline{Q}} \right\rceil$$

which is equivalent to

$$x(V \setminus S : S) \geq \left\lceil \frac{|S|}{\overline{Q}} \right\rceil . \qquad (4)$$

These inequalities are called generalized cut constraints.

Example 2: new inequalities

Take an ERMS inequality and divide by $\underline{Q}$. The rounded inequality is:

$$|S| - \left\lfloor \frac{|S|}{\underline{Q}} \right\rfloor \leq \left\lceil \frac{\underline{Q}-1}{\underline{Q}} \right\rceil x(S:S') + \left\lceil \frac{\underline{Q}-1}{\underline{Q}} \right\rceil x(S':S) + x(S:S)$$

which is equivalent to

$$x(S:S') + x(S':S) + x(S:S) \geq |S| - \left\lfloor \frac{|S|}{\underline{Q}} \right\rfloor . \qquad (5)$$

These inequalities are called rounded ERMS inequalities.

Rounded ERMS inequalities

By using the degree equations, the Rounded ERMS inequalities can be re-written as:

$$x(S \cup \{1\} : S \cup \{1\}) \leq |S| + \left\lfloor \frac{|S|}{\underline{Q}} \right\rfloor \qquad \forall S \subseteq V \setminus \{1\}.$$

Two interesting observations:

- The weaker linear inequalities

$$x(S \cup \{1\} : S \cup \{1\}) \leq |S| + \frac{|S|}{\underline{Q}}$$

  can be separated in polynomial time through a min-cut alg.
- When $|S| < \underline{Q}$ then the Rounded ERMS inequality says:

$$x(S \cup \{1\} : S \cup \{1\}) \leq |S|$$

Utility of the rounded ERMS inequalities

Fractional solution for an instance with $n = 8$, $\overline{Q} = 4$ and $\underline{Q} = 3$:

$$
\begin{array}{llllll}
x_{12} = 1 & x_{13} = 1/3 & x_{15} = 1/2 & x_{17} = 1/2 & x_{23} = 2/3 & x_{24} = 1/6 \\
x_{26} = 1/6 & x_{31} = 2/3 & x_{34} = 1/3 & x_{46} = 5/6 & x_{48} = 1/6 & x_{57} = 1/6 \\
x_{58} = 5/6 & x_{61} = 2/3 & x_{67} = 1/3 & x_{74} = 1/2 & x_{75} = 1/2 & x_{81} = 1
\end{array}
$$

This solution satisfies all MS inequalities (1), RMS inequalities (2) and rounded MS inequalities (4).

However, it violates the rounded ERMS inequalities (5) defined by the sets $S_1 = \{2, 3\}$ and $S_2 = \{2, 3, 4, 6\}$.

Definition

Consider now the CVRP where m is an input parameter.

The CVRP variant is a BVRP with $\underline{Q} = (|V| - 1) - (m - 1)\overline{Q}$.

Negative results

Proposition

For the CVRP with a fixed number of vehicles m, the condition $x(V \setminus S : S) \leq m - 1$ holds if and only if the MS inequality (1) for set S implies the ERMS inequality (3) for set S'.

Negative results

Proposition

For the CVRP with a fixed number of vehicles m, the condition $x(V \setminus S : S) \leq m - 1$ holds if and only if the MS inequality (1) for set S implies the ERMS inequality (3) for set S'.

Proposition

For the CVRP with a fixed number of vehicles m, the condition $x(1 : S) \leq m - 1$ holds if and only if the MS inequality (1) for set S implies the RMS inequality (2) for set S'.

Negative results

Proposition

For the CVRP with a fixed number of vehicles m, the condition $x(V \setminus S : S) \leq m - 1$ holds if and only if the MS inequality (1) for set S implies the ERMS inequality (3) for set S'.

Proposition

For the CVRP with a fixed number of vehicles m, the condition $x(1 : S) \leq m - 1$ holds if and only if the MS inequality (1) for set S implies the RMS inequality (2) for set S'.

Proposition

For the CVRP with a fixed number of vehicles m, the rounded ERMS inequality for set S' is implied by the rounded MS inequality for set S.

Positive result

Fractional solution for an instance with $n = 8$, $\overline{Q} = 4$ and $m = 2$ (hence $\underline{Q} = 3$):

$$\begin{array}{llllll}
x_{13} = 1/3 & x_{14} = 1/3 & x_{15} = 1/3 & x_{16} = 1/3 & x_{17} = 1/3 & x_{18} = 1/3 \\
x_{23} = 4/9 & x_{31} = 4/9 & x_{32} = 2/9 & x_{34} = 1/3 & x_{41} = 1 & x_{54} = 1/3 \\
x_{62} = 4/9 & x_{68} = 5/9 & x_{73} = 2/9 & x_{75} = 2/3 & x_{78} = 1/9 & x_{82} = 1/3
\end{array}$$

This solution satisfies all MS inequalities (1), rounded MS inequalities (4) and the degree equation on the depot.

However, it violates the ERMS inequality (3) defined by the set $S = \{2, 3, 4\}$.

eilA101: with versus without

Q	$\underline{Q}$	r-LB	r-time	LB	UB	tot-time	r-LB	r-time	LB	UB	tot-time	m
38	1	650.69	5.2	655.00	655	27.8	650.69	5.0	655.00	655	27.2	3
38	28	652.22	5.5	655.00	655	34.9	651.83	5.4	655.00	655	61.1	3
38	29	652.77	7.2	657.00	657	106.6	651.96	5.0	657.00	657	367.3	3
38	30	652.85	5.8	657.00	657	73.4	650.90	4.8	657.00	657	759.7	3
38	31	653.42	6.4	660.00	660	430.5	650.59	4.6	660.00	660	873.4	3
38	32	653.00	7.9	662.50	664	2 hours	651.03	5.1	656.63	665	2 hours	3
38	33	653.42	6.9	659.18	667	2 hours	651.48	4.8	655.00	672	2 hours	3
28	1	672.44	5.1	674.00	674	6.5	672.44	5.1	674.00	674	6.5	4
28	20	671.50	3.9	674.00	674	5.2	671.32	5.0	674.00	674	6.5	4
28	21	672.47	4.5	676.00	676	12.8	672.04	4.9	676.00	676	57.3	4
28	22	671.98	4.2	676.00	676	15.9	671.22	4.3	676.00	676	111.6	4
28	23	672.50	5.1	678.00	678	84.4	669.82	4.5	678.00	678	373.7	4
28	24	672.62	5.6	680.00	680	346.1	669.80	4.2	680.00	680	2800.1	4
28	25	675.08	7.7	684.00	684	3430.1	671.06	4.0	673.52	—	2 hours	4
23	1	695.02	5.6	704.00	704	273.6	695.02	5.5	704.00	704	270.3	5
23	12	697.00	4.4	704.00	704	142.3	695.00	5.1	704.00	704	288.5	5
23	13	697.73	7.0	704.00	704	100.3	694.26	4.5	704.00	704	885.9	5
23	14	697.31	4.8	704.00	704	197.7	694.17	4.7	704.00	704	1342.6	5
23	15	697.40	5.1	705.00	705	284.8	694.27	5.3	705.00	705	1623.0	5
23	16	698.48	5.2	705.00	705	222.4	694.65	5.6	705.00	705	1072.7	5
23	17	698.50	4.2	706.00	706	245.5	690.89	4.3	706.00	706	5640.3	5
23	18	699.50	4.1	707.00	707	188.3	694.75	4.9	698.74	736	2 hours	5
23	19	700.65	5.3	708.00	708	1323.3	689.11	5.0	697.12	730	2 hours	5
23	20	701.75	6.0	707.38	721	2 hours	688.37	4.6	693.84	—	2 hours	5

A071-03f: with versus without

Q	$\underline{Q}$	r-LB	r-tm	LB	UB	tot-time	r-LB	r-tm	LB	UB	tot-time	m
26	1	2030.00	1.3	2092.00	2092	45.5	2030.00	3.2	2092.00	2092	46.9	3
26	18	2035.70	1.7	2092.00	2092	49.3	2031.00	1.5	2092.00	2092	29.8	3
26	19	2037.43	1.9	2092.00	2092	17.4	2021.61	1.2	2092.00	2092	186.5	3
26	20	2041.00	1.7	2105.00	2105	63.5	2028.50	3.4	2105.00	2105	231.7	3
26	21	2037.50	1.0	2105.00	2105	76.7	2035.00	1.0	2105.00	2105	552.9	3
26	22	2037.50	1.5	2105.00	2105	183.0	2029.57	1.6	2105.00	2105	1001.6	3
26	23	2037.20	2.4	2109.00	2109	400.6	2032.70	1.1	2099.00	2109	2 hours	3
20	1	2142.50	1.3	2246.00	2246	1000.1	2142.50	1.3	2246.00	2246	949.8	4
20	11	2153.50	2.0	2246.00	2246	744.1	2148.00	0.8	2246.00	2246	1100.2	4
20	12	2153.75	1.0	2253.00	2253	777.1	2160.00	1.3	2253.00	2253	2522.1	4
20	13	2165.46	1.3	2253.00	2253	667.0	2142.50	0.8	2242.75	2263	2 hours	4
20	14	2157.25	1.7	2257.00	2257	931.6	2162.00	0.9	2242.00	2260	2 hours	4
20	15	2169.00	1.8	2258.00	2258	1068.4	2144.50	1.2	2232.21	2311	2 hours	4
20	16	2175.88	1.9	2262.50	2272	2 hours	2128.66	1.1	2224.00	2281	2 hours	4
20	17	2177.20	1.4	2283.00	2283	6608.3	2162.80	1.3	2205.25	2378	2 hours	4
16	1	2291.78	2.3	2406.00	2406	1371.4	2291.78	2.1	2406.00	2406	1262.9	5
16	8	2296.50	1.8	2406.00	2406	869.3	2281.12	1.7	2406.00	2406	3379.3	5
16	9	2284.50	1.2	2406.00	2406	715.3	2276.50	1.4	2406.00	2406	1389.5	5
16	10	2289.67	1.6	2406.00	2406	306.6	2267.53	1.4	2406.00	2406	6070.3	5
16	11	2298.38	1.8	2416.00	2416	624.6	2293.52	1.1	2416.00	2416	2024.7	5
16	12	2306.00	2.2	2416.00	2416	2789.4	2268.07	1.2	2357.08	2452	2 hours	5
16	13	2323.17	2.0	2443.00	2443	4205.8	2277.68	1.8	2330.84	–	2 hours	5
16	14	2335.27	1.9	2441.25	2485	2 hours	2269.27	1.4	2330.45	–	2 hours	5