Extended formulations, non-negative factorizations and randomized communication protocols

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- Prove a result of the type: *every* $\langle$*other condition*$\rangle$ *extended formulation for the perfect matching polytope has* $\langle$*certain size*$\rangle$

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What we do:

- Prove a result of the type: *every ⟨other condition⟩ extended formulation for the perfect matching polytope has ⟨certain size⟩*
- Develop a new way of *interpreting* extended formulations

Extension complexity

Getting started

$P \subseteq \mathbb{R}^d$ polytope

$Q \subseteq \mathbb{R}^e$ polyhedron &
$\pi : \mathbb{R}^e \to \mathbb{R}^d$ linear map | define *extension* of P if $\boxed{\pi(Q) = P}$

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$\mathrm{xc}(P) :=$ *extension complexity* of P

$:=$ min #facets in an extension of P

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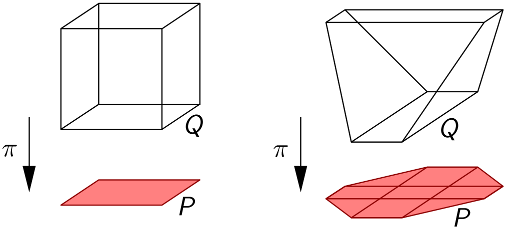
Example:

$\mathrm{xc}(\text{regular 8-gon}) \leqslant 6$



Extension complexity

More examples, and things for your to do list

Known results:

- xc(regular n-gon) $= \Theta(\log n)$ [BN01]
- xc(n-permutohedron) $= \Theta(n \log n)$ [G10]
- xc(spanning tree polytope of K_n) $= O(n^3)$ [M87]
- xc(stable set polytope of perfect graph G) $= n^{O(\log n)}$ [Y91]

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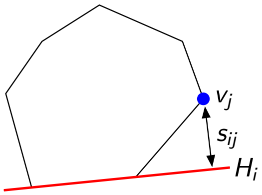
Open questions:

- xc(nonregular n-gon) $=$?
- xc(0/1 polytope P in $\mathbb{R}^d$) at most polynomial in d? [K]
- xc(perfect matching polytope of K_n) $=$?
- ...

Slack matrices

$$\begin{aligned} P &= \operatorname{conv}\{v_1, \dots, v_n\} \\ &= \{x \in \mathbb{R}^d : a_1^T x - b_1 \geqslant 0, \dots, a_m^T x - b_m \geqslant 0\} \end{aligned}$$

non-redundant inner/outer descriptions of P (assuming P full-dim.)



Definition

$S(P) :=$ *slack matrix* of P

$:= m \times n$ matrix (s_{ij}) with $s_{ij} = a_i^T v_j - b_i \geqslant 0$

$=$ slack of jth vertex w.r.t. ith facet

Non-negative rank

$S \in \mathbb{R}_{+}^{m \times n}$

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Definition

$$\begin{aligned} \mathrm{rank}_{+}(S) &:= \textit{non-negative rank of } S \\ &:= \min r \text{ s.t. } S = AB \text{ with } A \in \mathbb{R}_{+}^{m \times r} \text{ and } B \in \mathbb{R}_{+}^{r \times n} \\ &= \min r \text{ s.t. } S \text{ is sum of } r \text{ non-neg. rank 1 matrices} \end{aligned}$$

Non-negative rank... of a slack matrix

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Theorem (Y91, FKPT10+)

$$xc(P) = rank_{+}(S(P))$$

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Why like this result?

- elegant mathematical fact
- connects several research fields together
  - polyhedral combinatorics
  - applied linear algebra
  - communication complexity

Deterministic communication protocols

The tale of Alice and Bob

$f : X \times Y \to \{0, 1\}$ boolean function (matrix)

Two players:

- Alice knows $x \in X$
- Bob knows $y \in Y$

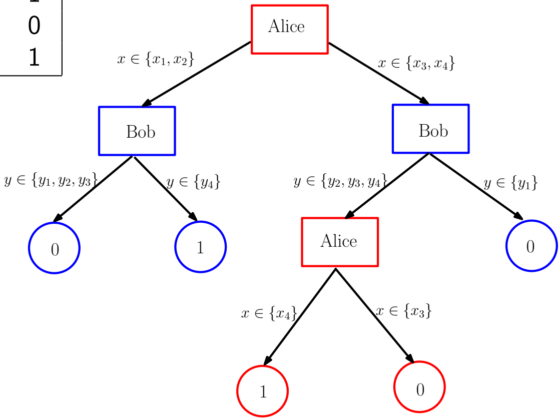
want to *compute* $f(x, y)$ *by exchanging bits*

Goal: Minimize *complexity* $:=$ #bits exchanged

Deterministic communication protocols

An example of a protocol

	y_1	y_2	y_3	y_4
x_1	0	0	0	1
x_2	0	0	0	1
x_3	0	0	0	0
x_4	0	1	1	1



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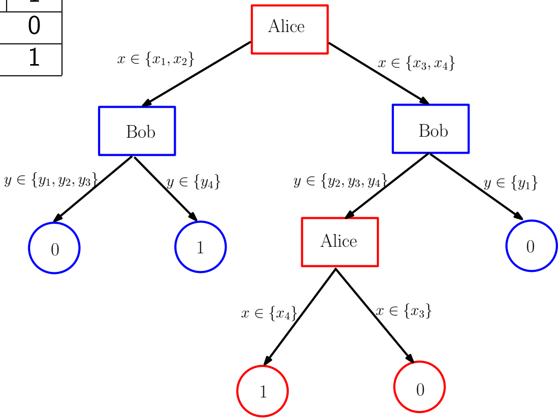

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Cases where Alice and Bob provide an extended formulation

Assume: $P = \mathrm{STAB}(G)$ for G perfect graph

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Theorem (Y91)

For all n-vertex perfect graphs G,

$$xc(STAB(G)) \leqslant 2^{O(\log^2 n)} = n^{O(\log n)}$$

Proof.

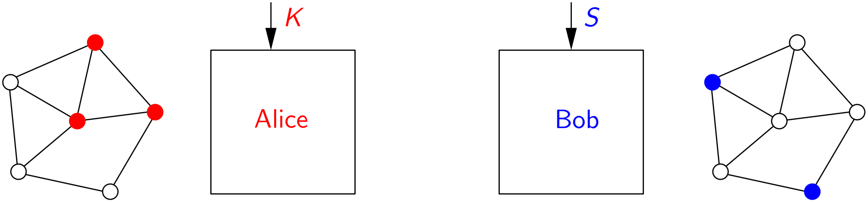
$\exists O(\log^2 n)$ complexity protocol for computing $S(\text{STAB}(G))$ $\square$

Deterministic communication protocols

Perfect without claw

Example: $P = \mathrm{STAB}(G)$ for G claw-free perfect graph

$\exists\ 3\log n + O(1)$ complexity protocol for $S(P) \implies \boxed{\mathrm{xc}(P) = O(n^3)}$



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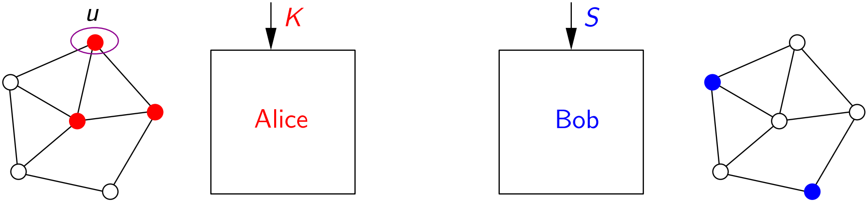

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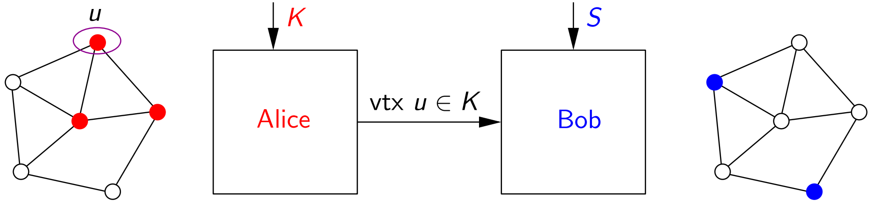

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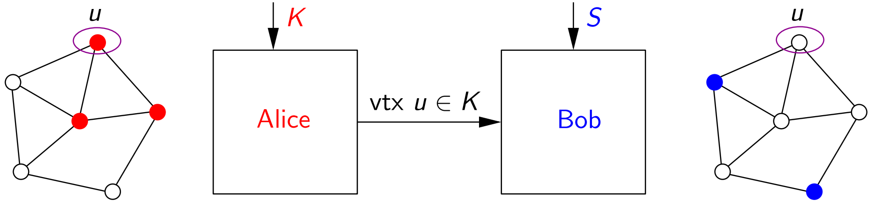

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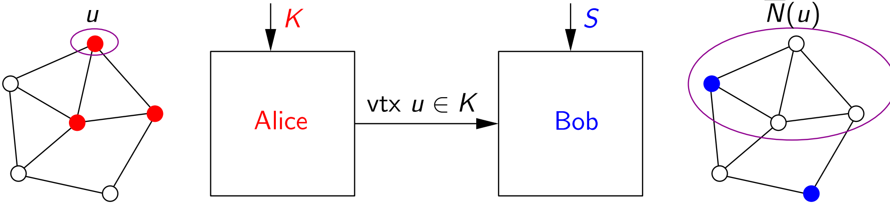

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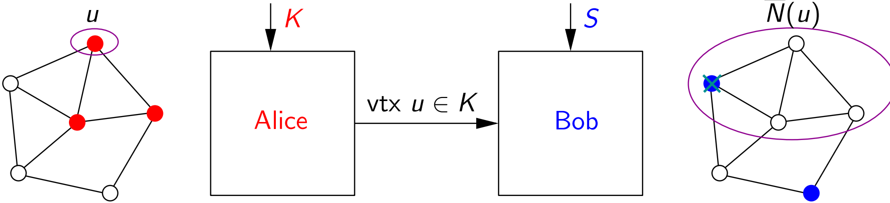

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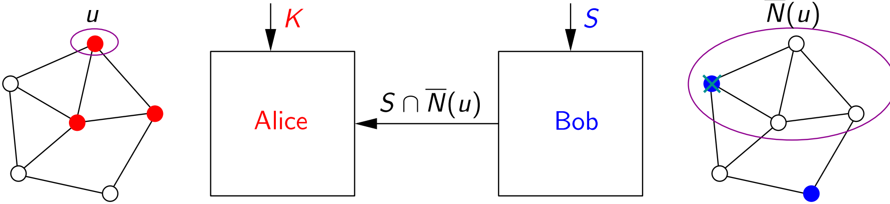

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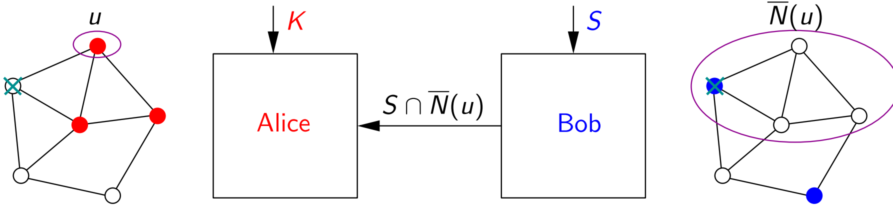

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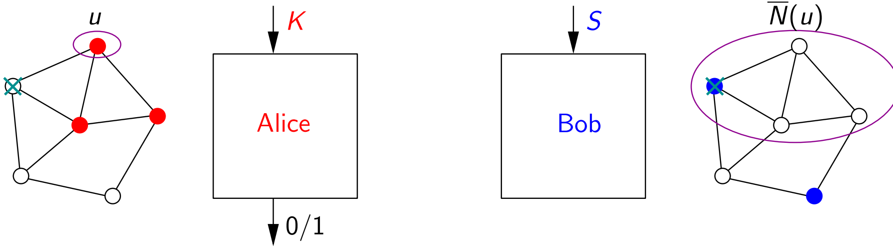

Deterministic communication protocols

How good are they?

Limitations:

1. binary slack matrices $S(P)$
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Possible solutions:

1. Alice and Bob could output non-binary values
2. randomized protocols would allow overlapping rectangles

Our main result

A tight relationship between extended formulations and communication complexity

Theorem

If c is the minimum complexity of a randomized communication protocol computing $S(P)$ on average, then

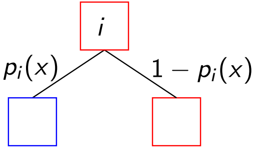
$$\boxed{xc(P) = rank_+(S(P)) = \Theta(2^c)}$$

Randomized communication protocols

Introducing the model

Similar to deterministic protocols except that

- Alice and Bob can use private random bits to make a choice



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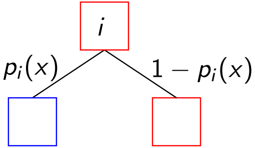

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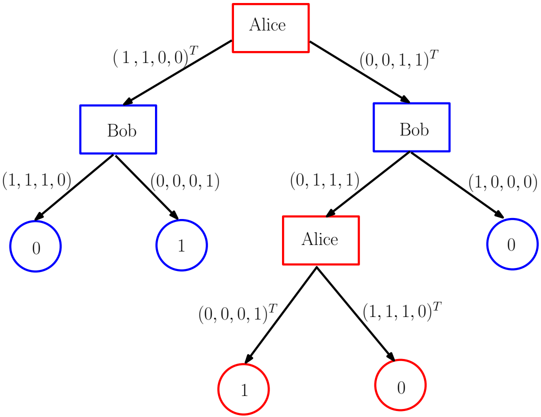
We say that the protocol *computes f on average* if: $\forall (x, y) \in X \times Y$,

$$E[\text{value output by the protocol on input } (x, y)] = f(x, y)$$

Randomized communication protocols

Go randomized

The previous deterministic protocol can be viewed as a randomized protocol where all the probabilities are either zero or one



In general can use arbitrary probabilities, and nonnegative values!

Proof of the theorem

Factorization $\implies$ protocol

Write $S(P) = TU$, where

- $T \in \mathbb{R}_+^{m \times r}$ row stochastic (w.l.o.g.)
- $U \in \mathbb{R}_+^{r \times n}$
- $r = \text{rank}_+(S(P)) + 1$

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Protocol:

- Alice gets row index i, Bob gets column index j
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Expected value on input (i, j): $\sum_{k=1}^{r} t_{ik} u_{kj} = s_{ij}$

Complexity: $\log \text{rank}_+(S(P)) + O(1)$

New lower bound for the perfect matching polytope of K_n

Now: $P =$ perfect matching polytope of K_n

Slack matrix: rows correspond to odd sets S
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slack of M w.r.t. S is $|M \cap \delta(S)| - 1$

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Theorem ("small randomness implies large size")

Consider an extension for P and a corresponding randomized protocol. If the probability that the protocol outputs a non-zero value, given a pair (S, M) with non-zero slack, is at least $p(n) \gg 1/n$, then the protocol has complexity $\Omega(np(n))$ and the extended formulation has size $2^{\Omega(np(n))}$

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Even more particular case: If the protocol is deterministic, then extension has exponential size

Proof sketch (reduction from SET DISJOINTNESS)

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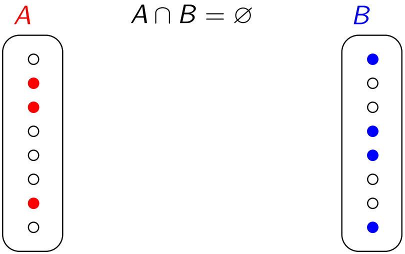

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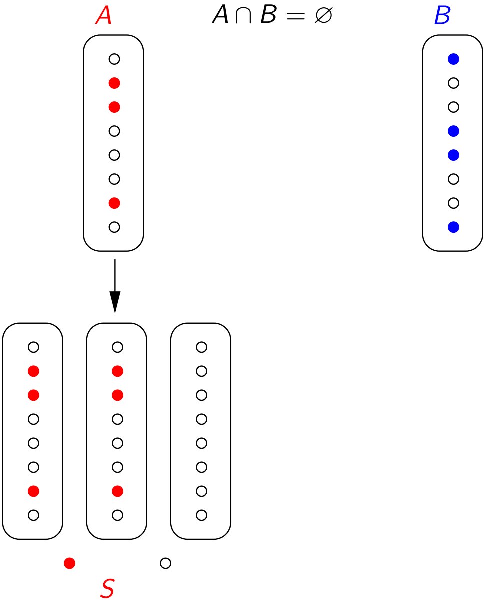

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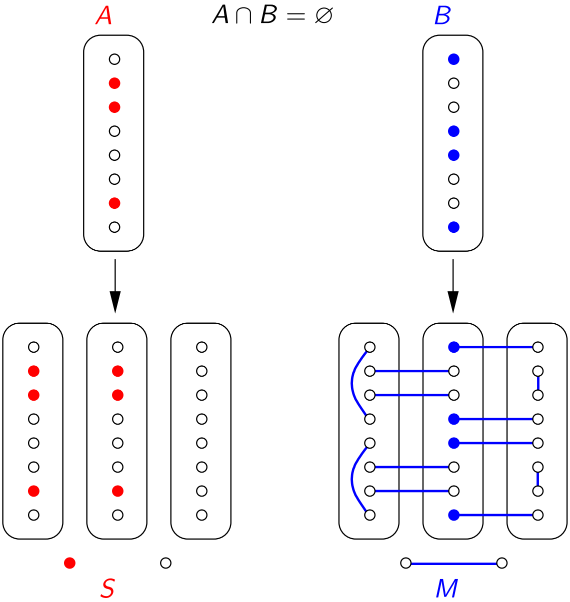

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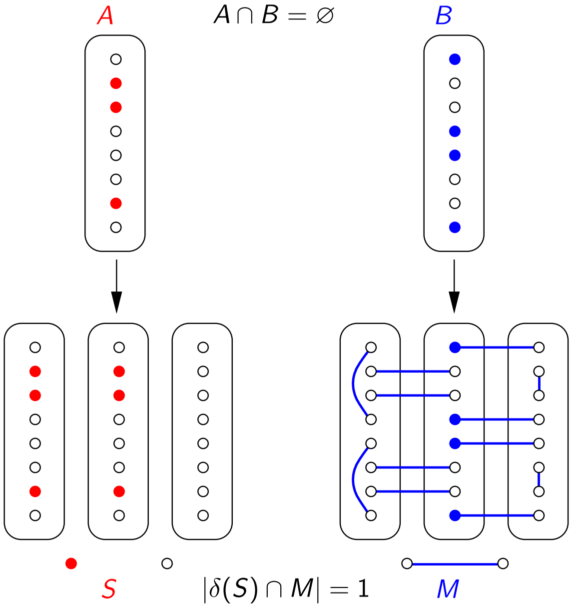

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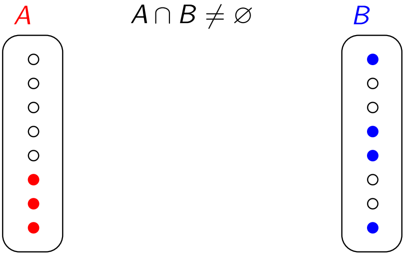

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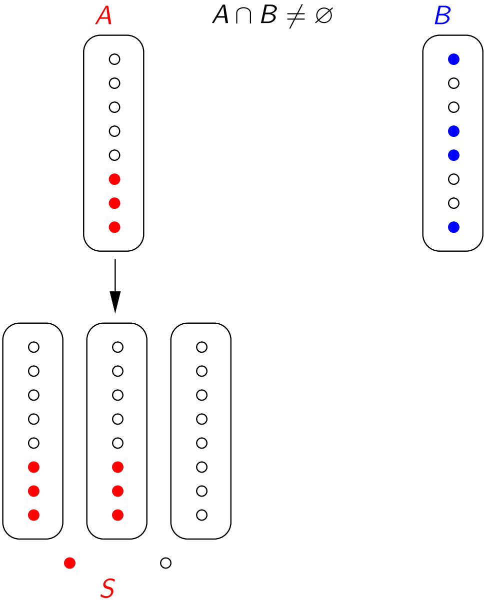

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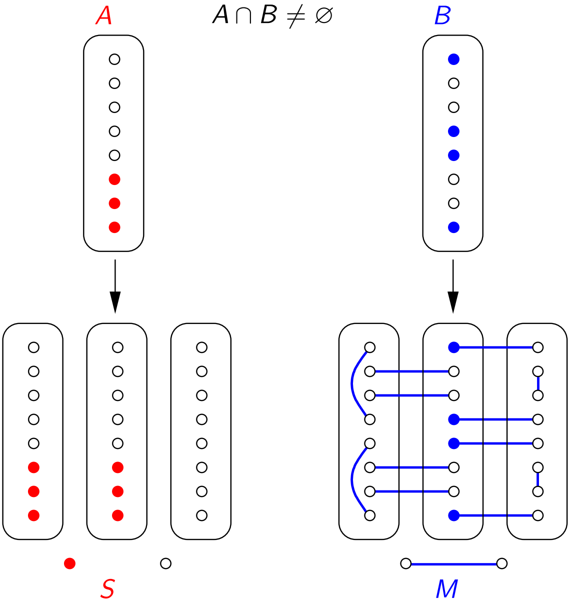

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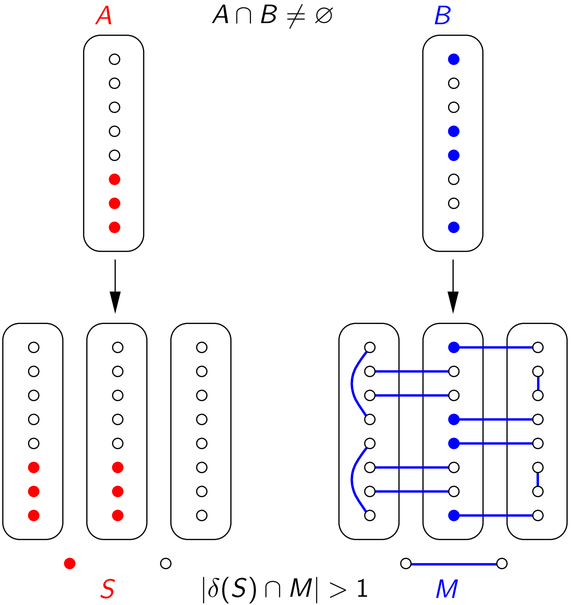

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Next,

- repeat protocol $\Theta(1/p(n))$ times on pair (S, M), independently
- declare A and B non-disjoint as soon as know that $\text{slack}(S, M) \neq 0$
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Finally, use

Fact. computing DISJ with high probability needs $\Omega(n)$ bits

Concluding remarks / questions

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- What is the communication complexity of the relation

$$\{(S, M, e) : S \text{ odd set}, M \text{ perfect matching}, e \in \delta(S) \cap M\} \text{ ?}$$

Concluding remarks / questions

- "Small randomness implies large size" also holds for spanning trees!!
- $\exists$ nice characterization of polytopes with binary slack matrix?
- What is the communication complexity of the relation

$$\{(S, M, e) : S \text{ odd set}, M \text{ perfect matching}, e \in \delta(S) \cap M\} \ ?$$

- ("log rank conjecture"-inspired) For which polytopes do we have

$$\text{randomized-cc}(\text{support of } S(P)) \leqslant \text{poly}(\log \text{rank}_+ S(P)) \ ?$$


Communication
Complexity
Eyal Kushilevitz and Noam Nisan

Thank You!