On convergence in mixed integer programming

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Integer programming - Gomory cuts

Definition

An inequality $cx \leq \lfloor \gamma \rfloor$ is a *Gomory cut* for a polyhedron P if $c \in \mathbb{Z}^m$ and if $cx \leq \gamma$ is valid for P.

Definition

The *Chvátal closure* P' of P is the set of all vectors that satisfy every Gomory cut for P.

Theorem (Chvátal, 1973, Schrijver, 1980)

For each rational polyhedron P, then

(i) *P' is again a rational polyhedron,*

(ii) *$P^{(k)} = P_I$ for some integer k,*

where $P^{(0)} := P$, and $P^{(i)} := (P^{(i-1)})'$, $\forall i \in \mathbb{N}$.

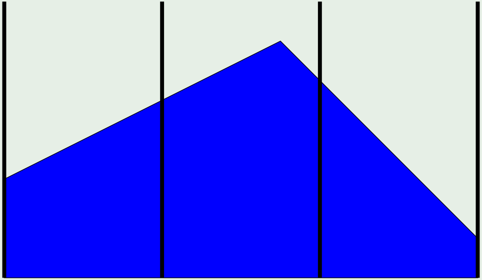
Split cuts

Definition

An inequality $cx + dy \leq \gamma$ is a *split cut* for $P \subseteq \mathbb{R}^{m+n}$ if there exists $(a, \beta) \in \mathbb{Z}^{m+1}$ such that $cx + dy \leq \gamma$ is valid for both

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \,:\, ax \leq \beta \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \,:\, ax \geq \beta + 1 \right\}.$$

Example



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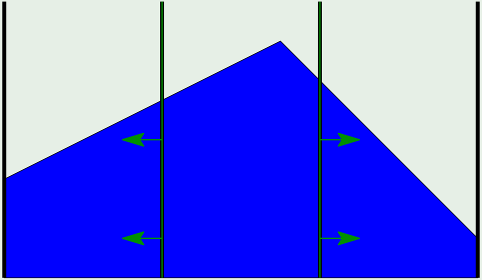

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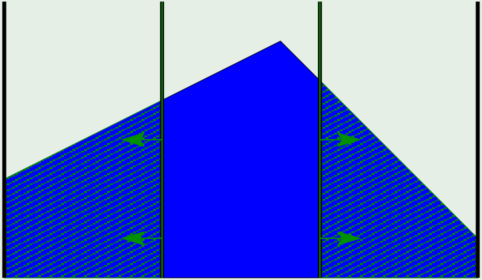

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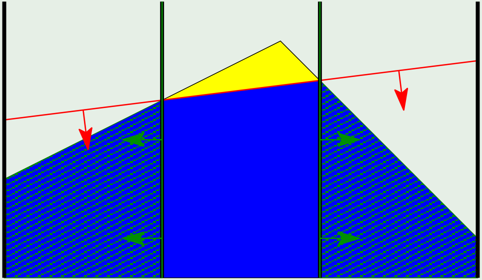


Split closure

Definition

The *split closure* $\mathcal{S}(P)$ of P is the set of all vectors that satisfy every split cut for P.

Theorem (Cook et al., 1990)

For each rational polyhedron P, then $\mathcal{S}(P)$ is again a rational polyhedron.

Definition

Let $\mathcal{S}^0(P) := P$, and $\mathcal{S}^i(P) := \mathcal{S}(\mathcal{S}^{(i-1)}(P))$, $\forall i \in \mathbb{N}$.

Observation (Cook et al., 1990)

- *If $P \neq P_I$, then $\mathcal{S}(P) \subsetneq P$;*
- *determining split closures does not suffice to generate P_I in a finite number of iterations.*

Finite convergence of split + rounding cuts

Theorem (Cook et al., 1990)

P_I can be generated in a finite number of iterations by combining split cuts with certain rounding cuts based on a discretization of the continuous variables.
(The discretization corresponds to a multiplication by the product of the subdeterminants of the initial constraint matrix corresponding to the continuous variables.)

We want to do it differently:

- in the original mixed integer setting, without discretizing the continuous variables;
- without remembering the original system.

Hausdorff convergence of the split closure

Theorem (Owen and Mehrotra, 2001)

For each polytope P, then $\lim_{i\to\infty} \mathcal{S}^i(P) = P_I$.

Theorem (Del Pia and Weismantel, 2010)

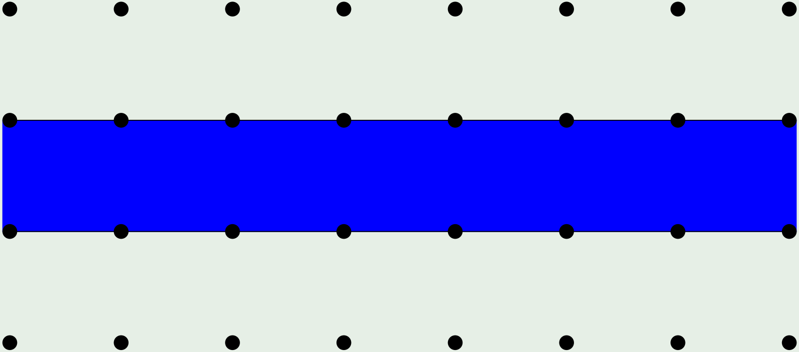
For each rational polyhedron P, then $\lim_{i\to\infty} \mathcal{S}^i(P) = P_I$.

Lattice-free sets

Definition

A polyhedron $L \subseteq \mathbb{R}^m$ is said to be *lattice-free* if $\mathrm{relint}(L) \cap \mathbb{Z}^m = \emptyset$.

Example



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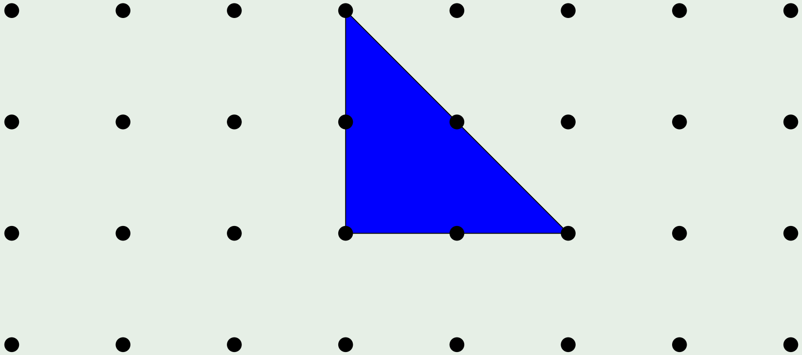

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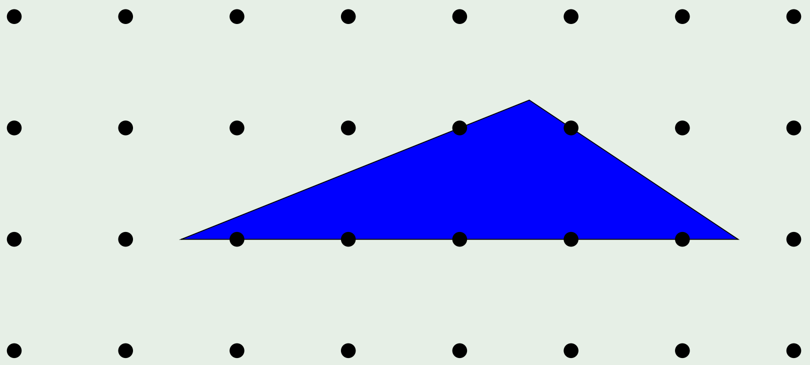

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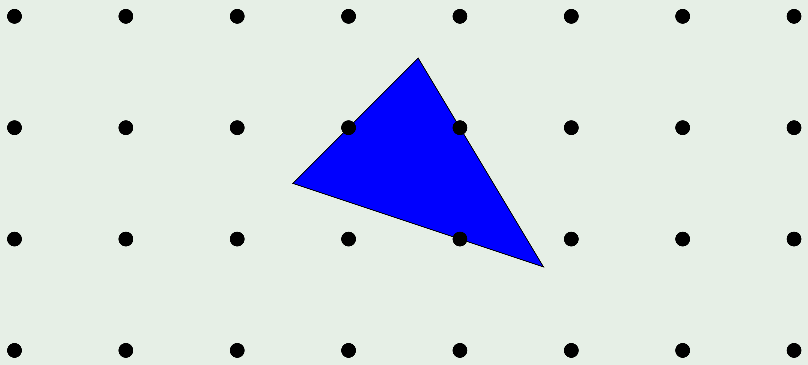

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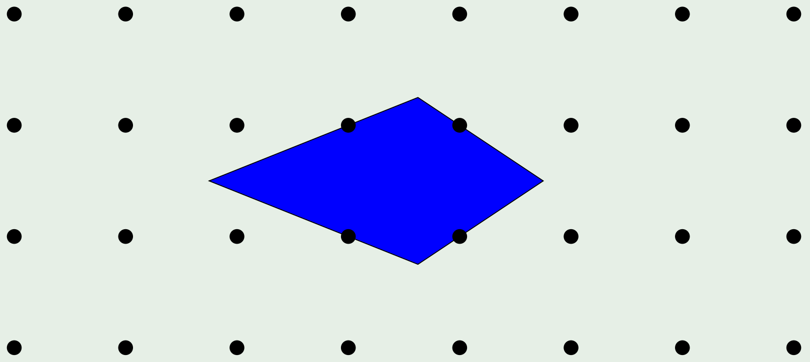


Integral lattice-free cuts

Definition

An inequality $cx + dy \leq \gamma$ is an *integral lattice-free cut* for $P \subseteq \mathbb{R}^{m+n}$ if there exists an integral lattice-free polyhedron $\{x \in \mathbb{R}^m : a_j x \leq \beta_j, j = 1, \ldots, k\}$ such that $cx + dy \leq \gamma$ is valid for every set

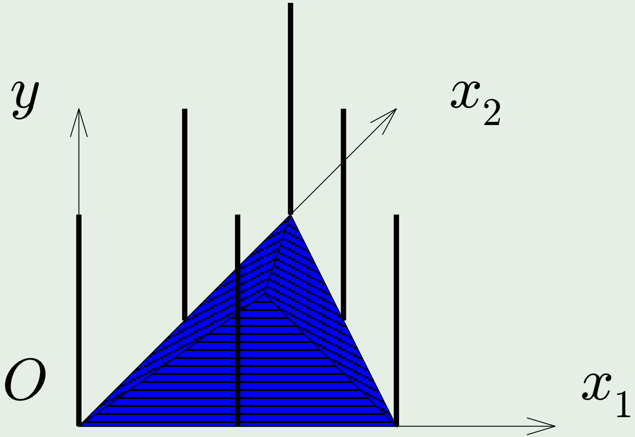
$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P : a_j x \geq \beta_j \right\}, \quad j = 1, \ldots, k.$$

Observation

A split cut is an integral lattice-free cut.

Integral lattice-free cuts

Example



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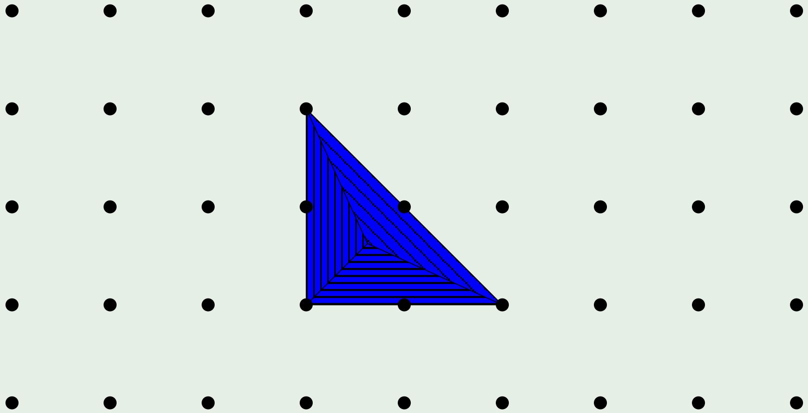

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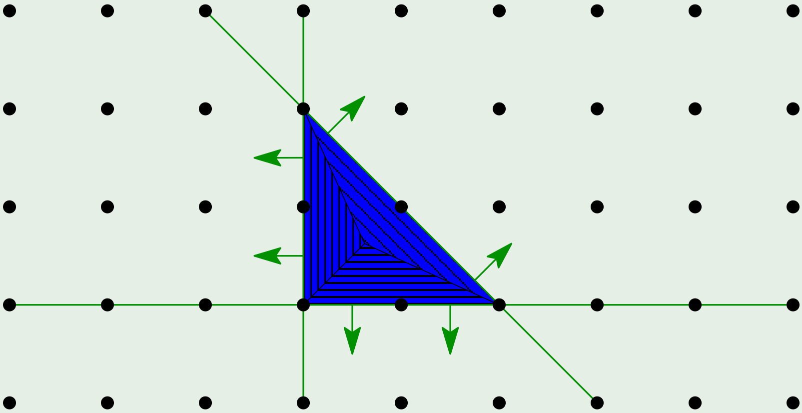

Integral lattice-free cuts

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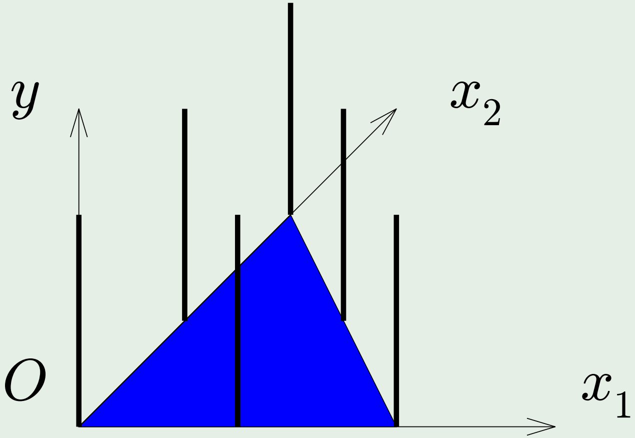

Integral lattice-free closure

Definition

The *integral lattice-free closure* $\mathcal{L}(P)$ of P is the set of all vectors that satisfy every integral lattice-free cut for P.

Theorem (Del Pia and Weismantel, 2010)

For each rational polyhedron P, then

(i) *$\mathcal{L}(P)$ is again a rational polyhedron,*

(ii) *$\mathcal{L}^k(P) = P_I$ for some integer k,*

where $\mathcal{L}^0(P) := P$, and $\mathcal{L}^i(P) := \mathcal{L}(\mathcal{L}^{(i-1)}(P)), \forall i \in \mathbb{N}$.

Proof sketch

Theorem (Del Pia and Weismantel, 2010)

For each rational polyhedron P, then $\mathcal{L}^k(P) = P_I$ for some $k \in \mathbb{N}$.

We show that for every vector $(c, d) \in \mathbb{Q}^{m+n}$ such that

$$\gamma := \max \left\{ cx + dy \;:\; \begin{pmatrix} x \\ y \end{pmatrix} \in P_I \right\}$$

is finite, there exists $k \in \mathbb{N}$ with

$$\max \left\{ cx + dy \;:\; \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{L}^k(P) \right\} = \gamma.$$

Proof sketch

Lemma

If the set

$$\operatorname{proj}_x \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P_I \ : \ cx + dy = \gamma \right\}$$

is not lattice-free, then $\exists k \in \mathbb{N}$ *with*

$$\max \left\{ cx + dy \ : \ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^k(P) \right\} = \gamma.$$

Proof sketch

- If $P_I = \emptyset$ the result follows from the convergence result for splits.
- We prove the theorem by induction on $\dim F$, where $F := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P_I \,:\, cx + dy = \gamma \right\}$.
- If F is a minimal face of P_I then the result follows from the previous Lemma.
- If $\mathrm{proj}_x F$ is not lattice-free, then the statement follows from Lemma.

Proof sketch

- Hence, we now assume that $\mathrm{proj}_x F$ is lattice-free.
- There exists an integral lattice-free polyhedron $L \subseteq \mathbb{R}^m$ of dimension m such that $\mathrm{proj}_x F \subseteq L$ and $\mathrm{relint}(\mathrm{proj}_x F) \subseteq \mathrm{relint}\, L$.
- There exists a $k \in \mathbb{N}$ such that

$$\mathrm{proj}_x \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{L}^k(P) \;:\; cx + dy > \gamma \right\} \subseteq \mathrm{relint}\, L.$$

- $\max \left\{ cx + dy \;:\; \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{L}^{k+1}(P) \right\} = \gamma.$

Open question

Do the two following sets coincide?

- Set of the integral polyhedra, among the maximal (wrt inclusion) lattice-free polyhedra;
- Set of the maximal (wrt inclusion) polyhedra, among the integral lattice-free polyhedra.

Observation

If they do coincide, then we have a necessary and sufficient class of lattice-free sets that gives us finite convergence to P_I.

Observation

If they do not coincide, then a necessary and sufficient class of lattice-free sets that gives us finite convergence to P_I *does not exist.*