Fixed-charge transportation on a path: linear programming formulations.

Mathieu VAN VYVE[1]

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Abstract

The fixed-charge transportation problem is a fixed-charge network flow problem on a bipartite graph. This problem appears as a subproblem in many hard transportation problems, and is also both a special case and a strong relaxation of the challenging big-bucket multi-item lot-sizing problem. In this paper, we provide a polyhedral analysis of the polynomially solvable special case in which the associated bipartite graph is a path.

We describe a new class of inequalities that we call "path-modular" inequalities. We give two distinct proofs of their validity. The first one is direct and crucially relies on sub- and super-modularity of an associated set function, thereby providing an interesting link with flow-cover type inequalities. The second proof is by showing that the projection of an $\mathcal{O}(n^2)$-size extended linear programming formulation onto the original variable space yields exactly the polyhedron described by the path-modular inequalities. Thus we also show that these inequalities suffice to describe the convex hull of the feasible solutions to this problem. We finally show how to solve the separation problem associated to the path-modular inequalities in $\mathcal{O}(n^3)$ time.

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[1]Université catholique de Louvain, CORE and LSM, B-1348 Louvain-la-Neuve, Belgium.

1 Introduction

In the fixed charge transportation problem (FCT), we are given a set of depots $i \in I$ each with a quantity of available items C_i, and a set of clients $j \in J$ each with a maximum demand D_j. For each depot-client pair (i, j), both the unit profit $q_{i,j}$ of transporting one unit from the depot to the client is known, together with the fixed charge $g_{i,j}$ of transportation along that arc. The goal is to find a profit-maximizing transportation program. Problem FCT can therefore be expressed as the following mixed-integer linear program:

$$\max \quad \sum_{i \in I} \sum_{j \in J} (q_{i,j} w_{i,j} - g_{i,j} v_{i,j}), \tag{1}$$

$$\sum_{j \in J} w_{i,j} \leq C_i, \qquad \forall i \in I \tag{2}$$

$$\sum_{i \in I} w_{i,j} \leq D_j, \qquad \forall j \in J \tag{3}$$

$$0 \leq w_{i,j} \leq \min(C_i, D_j) v_{i,j}, \qquad \forall i \in I, j \in J, \tag{4}$$

$$v \in \{0,1\}^{I \times J}, \tag{5}$$

where $w_{i,j}$ is a variable representing the amount transported from depot i to client j and $v_{i,j}$ is the associated binary setup variable.

In this description, the role of clients and depots are interchangeable. Indeed, this problem can be modelled as a bipartite graph in which nodes are either depots or clients and edges between a depot and a client exist if the client can be served from that depot. A standard variant (and indeed a special case) is that in which the demand of each client must be satisfied, in which case the unit profit is usually replaced by a unit cost.

Problem FCT can be considered as a basic problem in supply chain management in its own right, and is also a special fixed-charge network flow problem. However surprisingly few polyhedral results are known for FCT. When there is only one client or one depot, FCT reduces to a single node flow set for which the (lifted) cover and reverse cover inequalities have been described and shown to be effective [PvRW85, vRW87, GNS99]. Note that this also implies that FCT is NP-Hard. The flow structure of FCT is similar to that of the Capacitated Facility Location (CFL) problem, but this last problem has fixed cost for opening depots (nodes) as opposed to transportation (edges). Known valid inequalities are essentially flow cover type inequalities [Aar98, CFLP00]. FCT also appears to be both a special case and a strong relaxation of the multi-item big-bucket lot-sizing problem [V10] for which dual gaps are still typically large in practice [JD04, VW06, AM07]. This actually constitutes the initial motivation of this work.

In this paper, we study the polynomially solvable special case of FCT in which the associated bipartite graph is a path. FCTP is also a relaxation of FCT, and it can therefore be hoped that polyhedral results for FCTP will be helpful in solving the general case.

Let $[k, l]$ denote $\{k, k+1, \ldots, l\}$. For FCTP, we assume that we have $n+1$ depots and clients (or nodes) in total, indexed by the set $[0, n]$. We index depot-client pairs $(i-1, i)$ (or edges) by $i \in [1, n]$. Depots are represented by even nodes while clients are represented by odd nodes. The problem FCTP can be formulated as the following mixed-integer program:

$$\max \quad \sum_{i=1}^{n} p_j x_j - \sum_{i=1}^{n} f_j y_j, \tag{6}$$

$$x_i + x_{i+1} \leq a_i, \qquad \forall i \in [1, n-1], \tag{7}$$

$$0 \leq x_i \leq \min(a_{i-1}, a_i) y_i, \qquad \forall i \in [1, n], \tag{8}$$

$$y \in \{0, 1\}, \qquad \forall i \in [1, n], \tag{9}$$

where x_i is the amount transported between $i-1$ and i, y_i is the setup variable associated with x_i, and p_i and f_i are respectively the unit profit and the fixed cost of transportation between $i-1$ and i. We denote the set of feasible solutions to (7)–(9) by X^{FCTP}.

Program (7)–(9) can easily be recast into the framework introduced by Conforti et al. [CDSEW09] by operating the change of variables $x'_i = \frac{x_i}{M}$ with M large enough. Continuous variables (nodes) are then linked by a simple path with arcs having alternating directions. This graph trivially only contains a polynomial number of subpaths, so that FCTP admits a compact extended formulation and is therefore polynomially solvable. Parts of Section 2 can be seen as specializing the extended linear-programming formulation of Conforti et al. [CWZar] for FCTP. In addition, in the special case of FCTP we are able to describe a dedicated combinatorial optimization algorithm (see Van Vyve [V10]) and to project the extended formulation onto the original variable space (see Sections 2 and 3).

Recently, using the same framework, Di Summa and Wolsey [DSW10] aim at studying the mixed-integer set in which continuous nodes are linked by a bidirected path. This model subsumes FCTP. However they are only able to characterize the convex hull of the set for two special cases that do not subsume FCTP.

The rest of the paper is organized as follows. In Section 2 we give a linear-programming extended formulation for FCTP counting $\mathcal{O}(n^2)$ variables and constraints and a combinatorial characterization of the facet-defining inequalities defining its projection onto the original variable space. We first prove that the proposed formulation is indeed an extended formution of $conv(X^{FCTP})$. This is done by showing that the matrix associated to a subset of the constraints is a network flow matrix, as in [CWZar]. We then prove that the constraint matrix associated with the projection cone of this formulation is a totally unimodular matrix, thereby showing that the coefficients of x_i in facet-defining inequalities of $conv(X^{FCTP})$ have the consecutive ones property. Finally, fixing the coefficients of x_i to one of their $\mathcal{O}(n^2)$ possible values, the separation problem becomes separable, with all subproblems equivalent to matching problems in closely related bipartite graphs. This enables us to give a combinatorial characterization of a family of valid inequalities sufficient to describe $conv(X^{FCTP})$.

In Section 3, we introduce a new class of inequalities for FCTP that we call the "path-modular inequalities". We give a direct proof of their validity relying on sub- and super-modularity properties of an associated set-function. This makes an intersting link with flow cover inequalities of which the validity for the single-node flow set relies on submodularity of a similar set function. We then show that the path-modular inequalities are in fact identical to the inequalities obtained by projection. We also give an $\mathcal{O}(n^3)$ separation algorithm for the path-modular inequalities.

We conclude by discussing future research on the topic.

Notation. Throughout this paper we use the following notation: $\triangle$ denotes the symetric difference, $[k, l] = \{k, k+1, \ldots, l\}$, $N = [1, n]$, E and O denote the even and odd integers respectively, $\underline{e}_i$ denotes the unit vector with component i equal to 1 and all others components equal to 0, $\underline{1}$ denotes the all ones vector, and $\tilde{y}$ denotes the vector y with odd components being complemented (i.e. replaced by $1 - y_i$).

2 Compact linear programming formulation and projection

In this section, we give an extended linear programming formulation for FCTP of size $\mathcal{O}(n^2)$ variables and constraints. It can be seen as a specialization for FCTP of the results of Conforti et al. [CWZar]. In addition, we are able to give a combinatorial characterization of the extreme rays of the associated projection cone, and therefore a combinatorial description of inequalities sufficients to describe $conv(X^{FCTP})$.

We start with a definition, a lemma that we state without proof (it can be found in Van Vyve [V10]) and a couple of observations.

Definition 1 *For given $j \in [0, n+1]$, let $\bar{\alpha}_{i,j}$ for $i \in [0, n+1]$ be the unique solution of the following system of $n+2$ linear equations: $x_i + x_{i+1} = a_i$ for $i \in [0, n]$ and $x_j = 0$.*

Lemma 1 *Let x be an extreme point of X^{FCTP}. Then x_i takes its value in the set $\{\bar{\alpha}_{i,j}\}_{j=0}^{n+1}$.*

Observation 1 *$\bar{\alpha}_{i,j}$ satisfy the two following properties:*

$$\bar{\alpha}_{i,j} + \bar{\alpha}_{i+1,j} = a_i, \qquad \forall i \in [0, n], j \in [0, n+1], \tag{10}$$

$$\bar{\alpha}_{i,j} = \begin{cases} \bar{\alpha}_{i,0} + \bar{\alpha}_{0,j} & i \in E \\ \bar{\alpha}_{i,0} - \bar{\alpha}_{0,j} & i \in O \end{cases} \tag{11}$$

Let $(j_0, j_1, \ldots, j_{\bar{m}})$ be a permutation of a subset of $[0, n+1]$ that removes duplicate entries and sorts $\{\bar{\alpha}_{1,j}\}_{j=0}^{n+1}$ in increasing order: $\bar{\alpha}_{1,j_0} < \bar{\alpha}_{1,j_1} < \ldots < \bar{\alpha}_{1,j_{\bar{m}}}$. For fixed $i \in [1, n]$, it follows from (11) that the same permutation removes duplicate entries and sorts $\{\bar{\alpha}_{i,j}\}_{j=0}^{n+1}$ in increasing (resp. decreasing) order for i odd (resp. even). Defining

$$\bar{\beta}_{i,k} = \begin{cases} \bar{\alpha}_{i,j_k} & \text{if } i \in [1, n] \cap O,\ k \in [0, \bar{m}], \\ \bar{\alpha}_{i,j_{k-1}} & \text{if } i \in [1, n] \cap E,\ k \in [1, \bar{m}+1]. \end{cases}$$

and $\gamma_k = \bar{\beta}_{1,k} - \bar{\beta}_{1,k-1} > 0$ for $k \in [1, \bar{m}]$, (11) implies that

$$\begin{aligned} \bar{\beta}_{i,k} - \bar{\beta}_{i,k-1} &= \gamma_k, \text{for } i \in [1, n] \cap O,\ k \in [1, \bar{m}], \\ \bar{\beta}_{i,k} - \bar{\beta}_{i,k+1} &= \gamma_k, \text{for } i \in [1, n] \cap E,\ k \in [1, \bar{m}]. \end{aligned}$$

Consider now the following formulation in which the intended meaning is that $z_{i,k} = 1$ if x_i

takes a value at least $\bar{\beta}_{i,k}$ and 0 otherwise.

$$x_i = \bar{\beta}_{i,0} + \sum_{k=1}^{\bar{m}} \gamma_k z_{i,k} \qquad \forall i \in O, \tag{12}$$

$$x_i = \bar{\beta}_{i,\bar{m}+1} + \sum_{k=1}^{\bar{m}} \gamma_k z_{i,k} \qquad \forall i \in E, \tag{13}$$

$$y_i \geq z_{i,k} \qquad \forall i \in N, k \in [1, \bar{m}] : \bar{\beta}_{i,k} > 0 \tag{14}$$

$$z_{i,k} + z_{i+1,k} \leq 1, \qquad \forall i \in N, k \in [1, \bar{m}] \tag{15}$$

$$z_{i,k} = 0, \qquad \forall i \in N, k \in [1, \bar{m}] : \bar{\beta}_{i,k} > \min(a_{i-1}, a_i). \tag{16}$$

$$z_{i,k} = 1, \qquad \forall i \in N, k \in [1, \bar{m}] : \bar{\beta}_{i,k} \leq 0 \tag{17}$$

$$z_{i,k} \geq 0, \qquad \forall i \in N, k \in [1, \bar{m}]. \tag{18}$$

The next proposition shows that this is a correct formulation of FCTP.

Proposition 2 *For any $(x, y) \in \mathbb{R}^n \times \mathbb{Z}^n$, $(x, y) \in X^{FCTP}$ if and only if there exists z such that (x, y, z) is feasible in* (12)-(18).

Proof. We first show that for any $(x, y) \in X^{FCTP}$ there exists z such that (x, y, z) is feasible in (12)-(18). Assuming wlog that (x, y) is an extreme point of X^{FCTP}, let $z_{i,k} = 1$ if x_i takes a value at least $\bar{\beta}_{i,k}$ and 0 otherwise. That this choice of z satisfies constraints (12)-(18) is clear except for (15). Suppose $i \in O$ (the case $i \in E$ is similar). Because $\bar{\beta}_{i,k} + \bar{\beta}_{i+1,k} = \bar{\alpha}_{i,j_k} + \bar{\alpha}_{i+1,j_{k-1}} = \bar{\alpha}_{i,j_k} + \bar{\alpha}_{i+1,j_k} + \gamma_k = a_i + \gamma_k > a_i$, the relation $z_{i,k} + z_{i+1,k} > 1$ would imply $x_i + x_{i+1} > a_i$, a contradiction.

We now show the converse. First let us show that constraints (7) are implied by (12),(13),(15). Suppose i is odd (the other case is similar). By (10) and definition of γ, we know that $\beta_{i,0} + \beta_{i+1,\bar{m}+1} = (\beta_{i+1,\bar{m}} + \beta_{i,\bar{m}}) + (\beta_{i,0} - \beta_{i,\bar{m}}) = a_i - \sum_{k=1}^{\bar{m}} \gamma_k$. Therefore we can write

$$\begin{aligned} x_i + x_{i+1} =& \bar{\beta}_{i,0} + \sum_{k=1}^{\bar{m}} \gamma_k z_{i,k} + \bar{\beta}_{i+1,\bar{m}} + \sum_{k=1}^{\bar{m}} \gamma_k z_{i+1,k} \\ &= a_i - \sum_{k=1}^{\bar{m}} \gamma_k + \sum_{k=1}^{\bar{m}} \gamma_k (z_{i,k} + z_{i+1,k}) \\ &\leq a_i - \sum_{k=1}^{\bar{m}} \gamma_k + \sum_{k=1}^{\bar{m}} \gamma_k = a_i \end{aligned}$$

We finally show that (x, y, z) feasible in (12)-(18) and $x_i > 0$ implies $y_i > 0$. Observe that because of (17) and the fact that $\bar{\beta}_{i,k} = 0$ for some k, (12) can be rewritten as $x_i = \sum_{k=1:\bar{\beta}_{i,k}>0}^{\bar{m}} \gamma_k z_{i,k}$. Because of (18) and $\gamma_k > 0$ for all k, $x_i > 0$ if and only if $z_{i,k} > 0$ for some k. This implies $y_i > 0$ using (14). Hence (12)-(18) is a correct formulation of FCTP. ■

Proposition 3 *Formulation* (12)-(18) *is a linear programming extended formulation of $conv(X^{FCTP})$.*

We show that extreme points of (12)-(18) are integral. By the change of variables $z'_{i,j} = -z_{i,j}$ and $y'_i = -y_i$ for i even and $z'_{i,j} = z_{i,j}$ and $y'_i = y_i$ for i odd, each constraint (14) and (15) becomes a bound on a difference of two variables. The matrix associated to this modified constraint system is therefore a network flow matrix. The result follows. ■

A linear programming extended formulation automatically leads to a characterization of the convex hull of the solutions in the original space of variables through projection. Indeed, testing if a point (x^*, y^*) satisfying $y^* \leq \underline{1}$ belongs to $conv(X^{FCTP})$ is equivalent to testing whether the following LP in variables z is feasible:

$$
\begin{aligned}
\max \quad & 0, \\
& \sum_{k \in K_i} \gamma_k z_{i,k} = x_i^* \qquad \forall i \in N, && (\Delta_i) \\
& z_{i,k} \leq y_i^* \qquad \forall i \in N, k \in K_i, && (\delta_{i,k}) \\
& z_{i,k} + z_{i+1,k} \leq 1, \qquad \forall i \in [1, n-1], k \in K_i \cap K_{i+1} && (\rho_{i,k}) \\
& z_{i,k} \geq 0, \qquad \forall i \in N, k \in K_i,
\end{aligned}
$$

where $K_i = \{k \in [1, \bar{m}] : 0 < \bar{\beta}_{i,k} \leq \min(a_{i-1}, a_i)\}$. Through LP duality, this is equivalent to testing whether the following LP is bounded:

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} x_i^* \Delta_i + \sum_{i=1}^{n} \sum_{k \in K_i} y_i^* \delta_{i,k} + \sum_{i=1}^{n-1} \sum_{k \in K_i \cap K_{i+1}} \rho_{i,k}, \\
& \gamma_k \Delta_i + \delta_{i,k} + \rho_{i-1,k} + \rho_{i,k} \geq 0, \qquad \forall i \in N, k \in K_i \\
& \rho_{i,k} = 0, \qquad \forall i \notin [1, n-1] \text{ or } k \notin K_i \cap K_{i+1}, \\
& \delta, \rho \geq 0,
\end{aligned}
$$

Dividing the constraint by γ_k and rescaling $\delta_{i,k}$ and $\rho_{i,k}$ by γ_k, one obtains the equivalent LP

$$
\begin{aligned}
\min \quad & \sum_{i=1}^{n} x_i^* \Delta_i + \sum_{i=1}^{n} \sum_{k \in K_i} \gamma_k y_i^* \delta_{i,k} + \sum_{i=1}^{n-1} \sum_{k \in K_i \cap K_{i+1}} \gamma_k \rho_{i,k}, && (19) \\
& \Delta_i + \delta_{i,k} + \rho_{i-1,k} + \rho_{i,k} \geq 0, \qquad \forall i \in N, k \in K_i && (20) \\
& \rho_{i,k} = 0, \qquad \forall i \notin [1, n-1] \text{ or } k \notin K_i \cap K_{i+1}, && (21) \\
& \delta, \rho \geq 0, && (22)
\end{aligned}
$$

This is true if (x^*, y^*) satisfies

$$
\sum_{i=1}^{n} x_i^* \Delta_i + \sum_{i=1}^{n} \sum_{k \in K_i} \gamma_k y_i^* \delta_{i,k} + \sum_{i=1}^{n-1} \sum_{k \in K_i \cap K_{i+1}} \gamma_k \rho_{i,k} \geq 0
$$

for all extreme rays of the cone associated to the last constraint system. We will characterize sufficient inequalities to describe the polyhedron $conv(X^{FCTP})$ by characterizing these extreme rays. Because $x^*, y^*, \gamma, \delta, \rho \geq 0$, extreme rays with negative cost satisfy $\Delta \leq 0$. Hence we can normalize rays by assuming $\Delta_i \geq -1$ for all $i \in N$.

A first observation is that $\Delta_i < 0$ for consecutive i's (otherwise the ray is the sum of two other rays). The following result is less trivial.

Proposition 4 *The matrix associated to the constraint system* (20) *is totally unimodular.*

Proof. Variable $\delta_{i,k}$ appears only in one constraint and can therefore be neglected. We prove the result by proving that for any subset J of the columns of the matrix B under consideration, there exists a partition (J^-, J^+) of J such that $\sum_{j\in J^+} b_{i,j} - \sum_{j\in J^-} b_{i,j} \in \{-1, 0, 1\}$ for all rows i.

If none of the variables Δ_i belong to J then the remaining matrix satisfies the consecutive ones property and is TU. So we can assume the contrary and consider columns associated to Δ_i belonging to J in increasing order of i: $i_1 < i_2 < \dots$. We assign Δ_{i_1} to J^+. Then we assign Δ_{i_j} to the same partition as $\Delta_{i_{j-1}}$ if the parity of i_j and i_{j-1} are different and to the other partition if the parity is the same (in particular, consecutive columns are assigned to the same partition).

Note that having partitioned columns associated to Δ, the problem becomes separable in k. Consider $\rho_{i',k}$ belonging to J and let $j^<$ be the highest index such that $i_{j^<} \leq i'$ and let $j^>$ be the lowest index such that $i_{j^>} > i'$. In other words, $i_{j^<}$ and $i_{j^>}$ are the two closest columns Δ_i before and after i' belonging to J. At least one of them exists. We assign $\rho_{i,k}$ to

- the same partition as $\Delta_{i_{j^<}}$ if $i_{j^<}$ and i' have a different parity,
- the opposite partition to that of $\Delta_{i_{j^<}}$ if $i_{j^<}$ and i' have the same parity,
- the same partition as $\Delta_{i_{j^>}}$ if i' and $i_{j^>}$ have the same parity,
- the opposite partition to that of $\Delta_{i_{j^>}}$ if i' and $i_{j^>}$ have a different parity.

Observe that these rules cannot be contradictory in case both $j^<$ and $j^>$ exist because of the choosen partitioning of Δ_i.

We claim that this partitioning satisfies the desired property for row (20) for any given i, k. If Δ_i does not belong to j, then $\rho_{i-1,k}$ and $\rho_{i,k}$ are assigned to opposite partitions and the property holds. If Δ_i belongs to j, then $\rho_{i-1,k}$ and $\rho_{i,k}$ are both assigned to the opposite partition to that of Δ_i and again the property holds. ■

Corollary 5 *Normalized extreme rays of negative cost of* (19)-(22) *satisfy* $\Delta_i = -1$ *if* $i \in [l, l']$ *and* 0 *otherwise for some* $l, l' \in N$.

Therefore we can assume without loss of generality Δ fixed accordingly, and analyze optimal solutions of the LP (19)-(22) under this assumption. The following characterization will be sufficient for our purposes.

Proposition 6 *For given* (x^*, y^*) *and fixed* Δ *according to Corollary 5, the set of optimal solutions to* (19)-(22) *is the same for all vectors* y^* *that admit the same ordering when sorted in decreasing order of* $\tilde{y}^*$*. When* y^* *is such that this ordering is unique, the optimal solution is unique as well.*

Proof. Observe that for fixed Δ, the LP (19)-(22) is separable in k. Furthermore, for given each k, the problem is a problem of the form:

$$\min \quad \sum_{i=1}^{n} y_i^* \delta_i + \sum_{i=1}^{n-1} \rho_k, \tag{23}$$
$$\delta_1 + \rho_1 \geq 1, \tag{24}$$
$$\delta_i + \rho_{i-1} + \rho_i \geq 1, \qquad \forall i \in [2, n-1], \tag{25}$$
$$\delta_n + \rho_n \geq 1, \tag{26}$$
$$\rho_i = 0, \qquad \forall i \in Q, \tag{27}$$
$$\delta, \rho \geq 0, \tag{28}$$

where Q can be choosen to represent constraints (21). From Proposition 4, we know that optimal solutions can be assumed to be integral. From a graphical perspective, we have an undirected path of which each node i must be covered either by itself at cost y_i^* or by one of its incident edges at cost 1. Note that as $0 \leq y_i^* \leq 1$, we can assume that in optimal solutions all inequalities (24)-(26) will be tight. Indeed, if inequality i is not tight and $\Delta_i > 0$, we can decrease it by 1 without increasing the objective. If inequality i is not tight and $\Delta_i = 0$, then ρ_{i-1} or ρ_i is positive. Suppose ρ_i. Then we can decrease ρ_i by 1 and increase y_{i+1}^* by 1 without increasing the objective.

Therefore the same problem can be modelled as the more classical perfect matching problem in the following bipartite graph. The node set is $V = I \cup I'$ where $I = I' = [1, \bar{n}]$ and we index nodes in I (resp. I') by i (resp. i'). The edge set E includes edges (i, i') if $i = i'$ with cost y_i^* and edges $(i, i+1)$ and $(i', i'+1)$ for all $i, i' \in [1, \bar{n}-1] \setminus Q$ with cost $\frac{1}{2}$. Note that this graph is bipartite but not under the usual partition $E \subseteq I \times I'$. The two problems are equivalent because if edge $(i, i+1)$ is in the matching, then the edge $(i', i'+1)$ for $i' = i$ must also be in the matching and together they cost 1.

Elementary cycles in this bipartite graph are of the form $i, i+1, \ldots, j, j', j'-1, \ldots, i'$ and can therefore be unambigously denoted by $C_{i,j}$ for $i < j$. Consider a given perfect matching M of the graph (V, E) just defined. $C_{i,j}$ is an alternating cycle with respect to M if and only if the four following conditions hold:

(i) M does not contain an edge (k, k') with $i < k < j$,

(ii) if either (i, i') or (j, j') but not both belongs to M, then i and j are of the same parity,

(iii) if either both (i, i') and (j, j') or none of them belong to M, then i and j are of different parity.

(iv) there is no $k \in Q$ with $i \leq k < j$.

Let $C_{i,j}$ be such an alternating cycle and consider the perfect matching M' obtained by taking the symetric difference $M' = M \triangle C_{i,j}$. Denoting the cost of matching M by $c(M)$, the structure of the graph implies that:

$$c(M') = C(M) + \begin{cases} 1 - y_i^* - y_j^* & \text{if both } (i,i') \text{ and } (j,j') \text{ belong to } M, \\ y_i^* + y_j^* - 1 & \text{if neither } (i,i') \text{ nor } (j,j') \text{ belong to } M, \\ y_i^* - y_j^* & \text{if } (j,j') \text{ belongs to } M \text{ but not } (i,i'), \\ y_j^* - y_i^* & \text{if } (i,i') \text{ belongs to } M \text{ but not } (j,j'). \end{cases}$$

Combining this with the characterization of an alternating cycle above, we obtain that the set of optimal solutions (matchings) to (23)-(28) will be the same for all vectors y^* that admit the same orderings when sorted in decreasing order of $\tilde{y}^*$. When this ordering is unique the optimal matching is unique as well as taking the symetric difference with any elementary alternating cycle will strictly increase its cost.

For fixed Δ, subproblems k of (19)-(22) will only differ by Q in constraint (27). Hence the same is true for optimal solutions of (19)-(22). ■

Corollary 7 *Together with bounds $x_i \geq 0$ and $y_i \leq 1$ for $i \in N$, the following family of valid inequalities is sufficient to describe the convex hull of X^{FCTP}:*

$$\sum_{i=l}^{l'} x_i \leq \tau(\mathcal{L}) + \sum_{i=l}^{l'} \sigma(i, \mathcal{L}) y_i, \tag{29}$$

where $l, l' \in N$, $l \leq l'$, $\mathcal{L}$ is a permutation of $[l, l']$ and $\tau(\mathcal{L}) = \sum_{i=1}^{n-1} \sum_{k \in K_i \cap K_{i+1}} \gamma_k \rho_{i,k}$ and $\sigma(i, \mathcal{L}) = \sum_{k \in K_i} \gamma_k \delta_{i,k}$ for the unique optimal solution (δ, ρ) of (19)-(22) obtained when $\mathcal{L}$ is the unique permutation that sorts $\tilde{y}^*$ in decreasing order and Δ is fixed at $\Delta_i = -1$ for $i \in [l, l']$ and 0 otherwise.

3 Path-modular inequalities

In this section we derive a new family of inequalities that we call path-modular inequalities. Before showing that they are equivalent to inequalities (29), we give a more direct and insightful proof of their validity for FCTP. This proof relies on submodularity and supermodularity properties of the following set function. For a given set $S \subseteq [1, n]$ and vector $a \in \mathbb{R}_+^{n+1}$, let $\phi(S, a)$ be defined as

$$\phi(S, a) = \max \quad \sum_{i=1}^{n} x_j,$$

$$(7)$$

$$x_i \geq 0, \qquad \forall i \in S,$$

$$x_i = 0, \qquad \forall i \in N \setminus S.$$

For notational convenience, we will sometimes omit the argument a in $\phi(S, a)$ when we unambiguously refer to the original input data of the problem. Let $\rho_i(S) = \phi(S + i) - \phi(S)$ be the increment function of i at S. Clearly $\rho_i(S)$ is always nonnegative, but it is neither globally submodular ($\rho_i(S) \geq \rho_i(T)$ for all $S \subset T, i \notin T$) nor supermodular ($\rho_i(S) \leq \rho_i(T)$ for all $S \subset T, i \notin T$).

We now define the path-modular inequalities. Let L be a subinterval $[l, l']$ of N, let $L = O_L \cup E_L$ be the partition of L into odd and even numbers, and let $\mathcal{L} = (j_1, j_2, \ldots, j_{|L|})$ be a permutation of L. Let $O_L^{j_k} = \{j_1, \ldots, j_k\} \cap O_L$ and let $E_L^{j_k} = \{j_1, \ldots, j_{k-1}\} \cap E_L$. We call the the following inequality the $(l, l', \mathcal{L})$-path-modular inequality:

$$\sum_{i \in L} x_i \leq \phi(O_L) + \sum_{i \in E_L} \rho_i(O_L \cup E_L^i \setminus O_L^i) y_i - \sum_{i \in O_L} \rho_i(O_L \cup E_L^i \setminus O_L^i)(1 - y_i) \tag{30}$$

The following proposition essentially characterizes certain vectors y at which a given path-modular inequality is tight.

Proposition 8 *Let an interval $L = [l, l']$ and a permutation $\mathcal{L} = (j_1, j_2, \ldots, j_{|L|})$ of L be given. For each $k \in [0, |L|]$, there exists a point $(x^k, y^k) \in X^{FCTP}$ which is tight for the corresponding $(l, l', \mathcal{L})$-path-modular inequality, satisfying*

$$y^k = \sum_{i \in O_L} \underline{e}_i \triangle \sum_{1}^{k} \underline{e}_{j_k}$$

and at which the inequality reduces to $\sum_{i \in Y^k} x_i \leq \phi(Y^k)$ where $Y^k = \{i \in N : y_i^k = 1\}$.

Proof. The proof is by induction on k. For $k = 0$, $Y^0 = O_L$ and the path modular inequality reduces to $\sum_{i \in O_L} x_i \leq \phi(O_L)$. By definition of ϕ, there exists x such that this inequality is satisfied at equality.

So let us assume that the proposition is true for $k-1$ and $k>0$. If j_k is even, then the inequality reduces to $\sum_{i\in Y^k} x \leq \phi(Y^{k-1}) + \rho_{j_k}(Y^{k-1}) = \phi(Y^{k-1}+j_k) = \phi(Y^k)$. If j_k is odd, then the inequality reduces to $\sum_{i\in Y^k} x \leq \phi(Y^{k-1}) - \rho_{j_k}(Y^{k-1}-j_k) = \phi(Y^k - j_k) = \phi(Y^k)$. By definition of ϕ, there exists x such that this inequality is satisfied at equality. ■

In particular the previous proposition shows that all path-modular inequalities are tight at points with exactly either all odd or all even edges open.

The following proposition essentially tells us that ϕ exhibits supermodularity when we keep on opening edges of the same parity as the edge i under consideration. Conversely, it tells us also that ϕ exhibits submodularity when we keep on opening edges of the opposite parity compared to the edge i under consideration. Note that, broadly speaking, this is also the case for flow cover inequalities: in that case each edge is at an odd distance from any other one, so that the associated max-flow function is completely submodular. The technical and lengthy proof of this proposition can be found in [V10].

Proposition 9 *Let $S \subset T \subseteq N$ and $i \in N \setminus T$ be given.*

(i) If $(T\setminus S) \subseteq E$ and $i \in E$, then $\rho_i(T) \geq \rho_i(S)$.

(ii) If $(T\setminus S) \subseteq O$ and $i \in O$, then $\rho_i(T) \geq \rho_i(S)$.

(iii) If $(T\setminus S) \subseteq E$ and $i \in O$, then $\rho_i(T) \leq \rho_i(S)$.

(iv) If $(T\setminus S) \subseteq O$ and $i \in E$, then $\rho_i(T) \leq \rho_i(S)$.

We are now ready to prove the validity of the path-modular inequalities.

Proposition 10 *The path-modular inequalities* (30) *are valid for X^{FCTP}.*

Proof. Let a feasible point $(x,y) \in X^{FCTP}$ be given with $Y = \{i \in N : y_i = 1\}$. Let $L \subseteq N$ be a set of consecutive integers and let a permutation $\mathcal{L} = (j_1, j_2, \ldots, j_{|L|})$ of L be given. We show that the point (x,y) satisfies the corresponding path-modular inequality.

Let $\bar{E}^i_L = E^i_L \cap Y$ and $\bar{O}^i_L = O^i_L \cap Y$. The following relations hold

$$\begin{aligned}
\sum_{i\in L} x_i &\leq \phi(Y) \\
&= \phi(O_L) - \sum_{i\in O_L\setminus Y} \rho_i(O_L \cup \bar{E}^i_L \setminus \bar{O}^i_L) + \sum_{i\in E_L\cap Y} \rho_i(O_L \cup \bar{E}^i_L \setminus \bar{O}^i_L) \\
&\leq \phi(O_L) - \sum_{i\in O_L\setminus Y} \rho_i(O_L \cup \bar{E}^i_L \setminus O^i_L) + \sum_{i\in E_L\cap Y} \rho_i(O_L \cup \bar{E}^i_L \setminus O^i_L) \\
&\leq \phi(O_L) - \sum_{i\in O_L\setminus Y} \rho_i(O_L \cup E^i_L \setminus O^i_L) + \sum_{i\in E_L\cap Y} \rho_i(O_L \cup E^i_L \setminus O^i_L) \\
&= \phi(O_L) - \sum_{i\in O_L} \rho_i(O_L \cup E^i_L \setminus O^i_L)(1-y_i) + \sum_{i\in E_L} \rho_i(O_L \cup E^i_L \setminus O^i_L) y_i
\end{aligned}$$

where the first inequality is by definition of ϕ, the first equality is by definition of the increment function ρ and the fact that the set argument of each term (ordered according to the order $(j_1, j_2, \ldots, j_{|L|})$) differs from the previous term by exactly the element being incremented or decremented, the second inequality is an application of Corollary 9 (ii) for the first term and (iv) for the second term, the third inequality is an application of Corollary 9 (iii) for the first term and (i) for the second term, and finally the last equality holds because the added terms in the sum are null. ■

We now prove that the path-modular inequalities are sufficient to describe $conv(X^{FCTP})$ by showing that they are the same as the inequalities (29) obtained by projection.

Proposition 11 *Together with bounds $x_i \geq 0$ and $y_i \leq 1$ for $i \in N$, the path-modular inequalities are sufficient to describe the convex hull of X^{FCTP}.*

Proof. Let $l, l' \in N$, $l \leq l'$ and a permutation $\mathcal{L}$ of $[l, l']$ be given. Consider any of the $|l' - l + 2|$ points (x^k, y^k) of Proposition 8 for the ordering $(j_1, j_2, \ldots, j_n) = \mathcal{L}$. Such a point lies on the boundary of $conv(X^{FCTP})$, and therefore a separation algorithm that maximizes the violation will output a valid inequality that is tight at this point. Consider now the LP (19)-(22) with Δ fixed at $\Delta_i = -1$ for $l \leq i \leq l'$ and 0 otherwise. This LP actually determines a valid inequality of the form $\sum_{i=l}^{l'} x_i \leq \pi_0 + \sum_{i=l}^{l'} \pi_i y_i$ maximizing the violation. Now observe that the permutation $\mathcal{L}$ sorts $\tilde{y}^k$ in decreasing order for any k. Therefore the corresponding inequality (29) is tight at (x^k, y^k) for any k. By Proposition 8, this is also the case for the path-modular inequality associated to $(l, l', \mathcal{L})$. As these $|l' - l + 2|$ points are affinely independent, these two inequalities are identical. ■

We now turn to the question of separating a point (x^*, y^*) with $x_i^* \geq 0$ and $y_i^* \leq 1$ for all $i \in N$ from the polyhedron defined by the path-modular inequalities. It follows directly from Proposition 6 that the ordering maximizing the violation sorts $\tilde{y}^*$ in decreasing order. Moreover this ordering is independent of L.

It is explained in [V10] how preprocessing that can be carried out $\mathcal{O}(n^2)$ operations makes it possible to compute each coefficient of a given path-modular inequality in constant time. The next result follows.

Proposition 12 *The separation problem associated with the path-modular inequalities can be solved in $\mathcal{O}(n^3)$ time. The separation problem associated with the path-modular inequalities with $|L| \leq k$ can be solved in $\mathcal{O}(n^2 + k^2 n)$ time.*

4 Conclusion

This paper is a polyhedral analysis of the Fixed Charge Transportation problem on Paths (FCTP). We describe a new family of valid inequalities for FCTP that we call path-modular inequalities. We give a direct proof of their validity relying on sub- and super-modularity properties of an associated set function. We show that they are sufficient to describe the convex hull of solutions to FCTP by projecting a linear-programming extended formulation. We also show how to separate path-modular inequalities in $\mathcal{O}(n^3)$ time.

In an extended version of this paper [V10] we also characterize extreme points of FCTP, we give a combinatorial optimization algorithm, we present an alternative extended formulation, we report on computational experiment in solving FCTP using the different formulations and cutting planes presented, and show that a substantial number of facets of FCT are in fact path-modular inequalities of path-relaxations of FCT.

In our view, this work can be pursued in three main directions. The first one is trying to generalize path-modular inequalities to other graph structures. In particular, it would be nice to be able to describe a family of inequalities that subsumes path-modular (paths) and simple flow-cover inequalities (stars). The second one is to study how this work can help in going forward in the study of doubly-linked mixing sets [DSW10]. The third one is to use the present work to better solve general fixed charge transportation problems.

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References

[AM07] K. Akartunal and A.J. Miller. A Computational Analysis of Lower Bounds for Big Bucket Production Planning Problems. Available at Optimization Online, *http://www.optimization-online.org/DB_FILE/*2007/05/1668*.pdf*, 2007.

[Aar98] K. Aardal. Capacitated facility location: separation algorithms and computational experience. *Mathematical Programming*, 81:149–175, 1998.

[AMO93] R.K. Ahuja, T.L. Magnanti, and J.B. Orlin. *Network Flows.* Prentice Hall, Inc., Englewood Cliffs, New Jersey 07632, 1993.

[CDSEW09] M Conforti, M Di Summa, F Eisenbrand, and L.A. Wolsey. Network formulations of mixed-integer programs. *Mathematics of Operations Research*, 34:194–209, 2009.

[CFLP00] R. Carr, L. Fleischer, V. Leung, and C. Phillips. Strengthening integrality gaps for capacitated network design and covering problems. In *Proceedings of the 11th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 106–115. 2000.

[CWZar] M Conforti, L.A. Wolsey, and G. Zambelli. Projecting an extended formulation for mixed-integer covers on bipartite graphs. *Mathematics of Operations Research*, to appear.

[DSW10] M. Di Summa and L.A. Wolsey. Mixing sets linked by bidirected paths. Technical report, CORE, Université catholique de Louvain-la-Neuve, 2010.

[GNS99] Z. Gu, G.L. Nemhauser, and M.W.P. Savelsbergh. Lifted flow cover inequalities for mixed 0-1 integer programs. *Mathematical Programming A*, 85(3):439–467, 1999.

[JD04] R. Jans and Z. Degraeve. Improved lower bounds for the capacitated lot sizing problem with setup times. *Operations Research Letters*, 32:185195, 2004.

[KB77] J. Krarup and O. Bilde. Plant location, set covering and economic lotsizes: an $O(mn)$ algorithm for structured problems. In L. Collatz, editor, *Optimierung bei Graphentheoretischen und Ganzzahligen Probleme*, pages 155–180. Birkhauser Verlag, Basel, 1977.

[PvRW85] M.W. Padberg, T.J. van Roy, and L.A. Wolsey. Valid linear inequalities for fixed charge problems. *Operations Research*, 33:842–861, 1985.

[vRW87] T.J. van Roy and L.A. Wolsey. Solving mixed integer programming problems using automatic reformulation. *Operations Research*, 33(1):45–57, 1987.

[V10] M. Van Vyve. Fixed-charge transportation on a path: optimization, LP formulations and separation. CORE Discussion Paper 2010/68, Université catholique de Louvain-la-Neuve, 2010.

[VW06] M. Van Vyve and L.A. Wolsey. Approximate extended formulations. *Mathematical Programming B*, 105(2):501–522, 2006.