Reverse Multistar Inequalities and Vehicle Routing Problems with lower bound capacities

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Abstract

This paper concerns a vehicle routing problem where each vehicle has upper and lower capacities. We assume one depot, an homogeneous fleet of vehicles and all customers requiring the same demand of a product. A feasible solution for this problem requires that all routes should visit a number of customers within a given interval specified by the lower and upper capacities. The problem is called *Balanced Vehicle Routing Problem* (BVRP). In this paper we adapt the single-commodity-flow formulation known in the literature for the so-called Capacitated Vehicle Routing Problem (CVRP). We also discuss some new inequalities that are irrelevant for the CVRP but which play a fundamental role for the BVRP. These are the so-called *Reverse Multistar inequalities* that are related to the *Multistar inequalities*, well known from the literature on the CVRP. This paper also proposes other new families of inequalities and analyzes computational experiments that show the convenience of using these new inequalities when solving BVRP instances. The experiments are based on variations of CVRP instances with up to 100 customers. In addition this paper studies the impact of using the new inequalities for solving CVRP instances where the number of vehicles is fixed. Although empirically these inequalities do not always reduce the computational time to solve larger instances when compared to other approaches in the literature, the paper analyzes several theoretical properties and motivates the idea that lower bound information might be useful for some other variations of the CVRP.

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1 Introduction

Several variations of routing problems arise in real-world applications. The basic problem is the one of designing a set of minimum-cost routes originating and finishing at a given location. This problem has a depot (where several capacitated vehicles are located) and a set of customers, each one requiring a known demand of a commodity which is available at the depot. The objective is to design routes with minimum length (or cost) to serve the customers. Applications impose several constraints on this basic problem. Among others we may consider:

1. capacity constraints stating that each vehicle cannot carry more than a certain number of goods;

2. distance constraints stating that the length of the route of each route cannot exceed a given length;

3. time windows stating that each client cannot be visited after a certain time;

4. precedence constraints requiring that a given customer can be visited only after another has been visited.

Here we study a variant which considers a lower limit on the capacity of each vehicle to simply state that a vehicle can be used only if it visits, at least, a certain number of clients. Similar constraints have already been considered in, for example, Groër, Golden and Wasil [7] and Jozefowiez, Semet and Talbi [11] where the routes of a solution must be balanced.

Single Commodity Flow (SCF) formulations provide a basic framework for modelling most of these problems (see, for instance, Gavish and Graves [5], Letchford and Salazar-González [13] and Toth and Vigo [15]). The SCF model is an example of an extended model, where there are two set of mathematical variables. One set is composed by the decision variables describing the design of the routes. The other set is composed by flow variables to capture the requirements on the customer demands and on the vehicle capacities. Given a SCF model, we can use a tool of projection to create a natural model based only on the route-design variables. Although the linear programming (LP) relaxation bounds from both models coincide and the natural models require cutting-plane generation approaches, they are very successful for solving some routing problems (see, e.g., Naddef and Rinaldi [14]).

In this paper the tool of projection is a theorem by Hoffman [10] and the projected inequalities are analyzed. The main messages can be summarized in five key items:

(i) The set of projected inequalities can be divided into two sets which exhibit different modeling properties. For many vehicle routing problems, including the standard *Capacitated Vehicle Routing Problem* (CVRP), one set contains redundant inequalities while the other contains inequalities that are known to be facet defining under mild conditions. This is the topic of Section 2. This classification raises the question of knowing whether the redundant set is redundant "for all vehicle routing problems".

(ii) The set of apparently non-interesting constraints become interesting for a less studied variation of the problem, namely the case where arc lower bound capacities are involved. This is the topic of Section 3, where we introduce the so-called *Balanced Vehicle Routing Problem* (BVRP), a particular case of the Balanced Billing Cycle Vehicle Routing Problem (see [7]).

(iii) We also show that in contrast with the other set of projected inequalities, the apparently non-interesting inequalities can be lifted, leading to a stronger set. This is also discussed in Section 3.

(iv) Following what is known for other inequalities, we shall also show that another related, relevant and quite intuitive set of inequalities can be obtained by division and rounding of coefficients. In Section 4 we introduce these rounded inequalities and, by presenting a simple example, show that the new inequalities might be useful to solve BVRP instances in a cutting plane fashion. Computational results presented in Section 6 give empirical evidence of the usefulness of these new inequalities on instances with up to 100 customers and unit demands.

(v) Finally, we point out that a variation of the standard CVRP imposes an implicit lower bound capacity on the vehicles, and thus inequalities proposed in (ii), (iii) and (iv) might be of interest to solve this variation. This is the topic of Section 5, with some experiments analyzed at the end of Section 6.

2 Multistar and Reverse Multistar Inequalities

Routing problems are usually modeled through a directed graph $G = (V, A)$. A special node in $V = \{1, 2, \ldots, n\}$, node 1, represents a depot. Nodes in $V \setminus \{1\}$ represent customers. Each node i has a demand d_i such that

$$\sum_{i \in V} d_i = 0.$$

Note that, since at least a node has a positive demand, at least another node has a negative demand. In CVRP all customers have positive demands and only the depot has a negative demand. In pickup-and-delivery routing problem (see, e.g., Hernández and Salazar [9]) several customers may have negative demands.

An arc is represented by one index a, or by two indices ij when its head j and its tail i are convenient for the notation. Each arc $a \in A$ is associated with a lower capacity $\underline{q}_a$ and an upper capacity $\overline{q}_a$ such that $\underline{q}_a \leq \overline{q}_a$, meaning that if arc a is in the solution (that is, used by a vehicle) then the vehicle load when traversing this arc cannot be greater than $\overline{q}_a$ and not smaller than $\underline{q}_a$. Also, each arc a is associated with a value c_a representing the cost of using the arc by a vehicle. A generic SCF model uses two variables for each arc a:

(i) a design binary variable x_a indicating whether arc a is used by a vehicle and

(ii) a continuous variable f_a representing the load (flow) of a vehicle traversing the arc.

To simplify notation, if $S, T \subset V$ then we write $x(S : T)$ instead of $\sum_{(i,j) \in A, i \in S, j \in T} x_{ij}$. Given $S \subseteq V \setminus \{1\}$, we denote $V \setminus (S \cup \{1\})$ by S'. For brevity of notation, we also write i instead of $\{i\}$ for any $i \in V$. In addition, $\delta^+(S)$ stands for $\{(i,j) \in A : i \in S, j \notin S\}$ and $\delta^-(S)$ stands for $\{(i,j) \in A : i \notin S, j \in S\}$.

Then, the model minimizes a cost function

$$\min \sum_{a \in A} c_a x_a$$

subject to the following two sets of constraints.

One set involves only the design binary variables, and imposes the in-degree and out-degree constraints on each customer node:

$$x(i : V \setminus \{i\}) = x(V \setminus \{i\} : i) = 1 \qquad \text{for all } i \in V \setminus \{1\} \tag{1}$$

$$x_a \in \{0, 1\} \qquad \text{for all } a \in A. \tag{2}$$

The other set of constraints imposes the connectivity and the capacity constraints, and involves the use of the continuous variables:

$$\sum_{a\in\delta^-(i)} f_a - \sum_{a\in\delta^+(i)} f_a = d_i \qquad \text{for all } i \in V \tag{3}$$

$$\underline{q}_a x_a \le f_a \le \overline{q}_a x_a \qquad \text{for all } a \in A. \tag{4}$$

As one specific example, a model for the unit-demand CVRP with vehicle capacity $Q \geq 2$ can be defined by setting

$$d_i := \begin{cases} 1 & \text{if } i \in V \setminus \{1\} \\ 1 - |V| & \text{if } i = 1 \end{cases}$$

and

$$\begin{aligned} \underline{q}_a = \overline{q}_a = 0 &\qquad \text{for all } a \in \delta^-(1) \\ \underline{q}_a = 1 \text{ and } \overline{q}_a = Q &\qquad \text{for all } a \in \delta^+(1) \\ \underline{q}_a = 1 \text{ and } \overline{q}_a = Q - 1 &\qquad \text{for all } a \notin \delta^+(1) \cup \delta^-(1). \end{aligned}$$

Different variations of vehicle routing problems can be formulated by changing some of the parameters given in the previous generic SCF formulation or setting new ones (see, e.g., Toth and Vigo [15]). The SCF model is an example of compact model since it involves a number of variables and constraints that is bounded by a polynomial function defined on the size (number of nodes and number of arcs) of the input instance. We can create a natural model involving only the x_a variables and with a LP relaxation bound equal to the LP relaxation bound of the SCF model, by using the tool of projection. The procedure to project out the continuous variables from the LP relaxation of the SCF model is based on the following result:

Theorem 2.1 (Hoffman 1960 [10]) *Given x_a values, there is a solution of the linear system (3)–(4) on the f_a variables if and only if*

$$\sum_{a\in\delta^-(S)} \overline{q}_a x_a \ge \sum_{a\in\delta^+(S)} \underline{q}_a x_a + \sum_{i\in S} d_i \qquad \textit{for all } S \subset V. \tag{5}$$

We can give some intuition on how to generate these inequalities from the SCF model. Suppose we add the flow conservation constraints (3) for all nodes i in a set S and cancel equal terms, leading to:

$$\sum_{a\in\delta^-(S)} f_a = \sum_{a\in\delta^+(S)} f_a + \sum_{i\in S} d_i.$$

Then, by using the upper bounding part in constraints (4) on the term in the left-hand side of the previous equality and by using the lower bounding part of constraints (4) on the right-hand side, we obtain (5). In fact, by reversing the bounding procedure just suggested (i.e., using the lower bounding part in (4) on the term in the left-hand side, and using the upper bounding part in (4) on the term in the right-hand side) we obtain the following alternative necessary-and-sufficient condition for the theorem

$$\sum_{a \in \delta^-(S)} \underline{q}_a x_a \leq \sum_{a \in \delta^+(S)} \overline{q}_a x_a + \sum_{i \in S} d_i \qquad \text{for all } S \subset V. \tag{6}$$

One can easily see that the inequality (6) associated with S coincides with the inequality (5) associated with the set $V \setminus S$. This follows from the fact that $\delta^+(S) = \delta^-(V \setminus S)$, $\delta^-(S) = \delta^+(V \setminus S)$ and that $\sum_{i \in S} d_i + \sum_{i \in V \setminus S} d_i = 0$. Thus, the two families of inequalities (5) and (6) are equivalent.

However, the two families of inequalities (5) and (6) (together with the way we suggested above for generating these inequalities) permit us to cast Hoffman's theorem in an alternative form. We can divide each family of inequalities in two groups: one group is associated with the sets S containing node 1 and the other group with the sets S not containing node 1. Then we use one group in both families of inequalities to generate a complete projection. These groups are the sets of inequalities (5) and (6) defined by sets S not containing node 1.

The reason for this option is that then it becomes easier to enhance the different modeling properties of the inequalities from each group. First, for most of the standard vehicle routing problems, the first group of projected inequalities (given by (5) for sets S not containing node 1) is quite interesting while the second group (given by (6) for sets S not containing node 1) is easily seen to be redundant. Second, expression (6) permits us to detect quite easily less standard variations of vehicle routing problems where these inequalities might be useful.

We use the name *Multistar* (MS) inequalities for constraints (5) with $1 \notin S$, while the name *Reverse Multistar* (RMS) inequalities is used for constraints (6) with $1 \notin S$. An intuition for the designation "reverse" will be given later on.

As noted before, it is well known (see, for instance, Gouveia [6], Letchford, Eglese and Lysgaard [12] and Letchford and Salazar-González [13]) that for "capacitated" routing problems, that is, routing problems with an upper bound on the number of customers on each route (or more generally, an upper bound on the sum of the demands of the customers on each route), the resulting MS inequalities turn out to be rather interesting inequalities. However, for these problems, the RMS inequalities are not of interest since they are implied by other inequalities in the model.

To exemplify this, consider again the unit-demand CVRP. The resulting MS inequalities are as follows

$$Qx(1:S) + (Q-1)x(S':S) \geq x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}. \quad (7)$$

They are the directed version of known inequalities that define facets of the associated undirected polytope (see Araque et al. [1]).

The resulting RMS inequalities for the unit-demand CVRP are given as follows

$$x(1:S) + x(S':S) \leq (Q-1)x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}.$$

It is easy to see that for a given set S the corresponding RMS inequality is implied by the equality obtained by adding the in-degree constraints (1) for nodes $i \in S$. Thus, for the CVRP, the RMS inequalities are not of interest. The next section presents and discusses a variant of the CVRP where the RMS inequalities are not redundant, and they have a specific and intuitive interpretation.

3 Vehicle routing with lower capacities

As we have noted before, expression (6) permits us to "guess" situations where the RMS inequalities might be of interest. The first situation that comes to mind is one where the demand summation on the right-hand side of (6) is negative. This may happen, for instance, in situations where some of the customer demands are negative. These situations arise in pick-up and delivery variations of the CVRP (see, e.g., [4]) where typically the customers with positive demands correspond to locations which receive some commodity from the depot and where customers with negative demands send some commodity to the depot. It is not difficult to see that there are interesting MS inequalities as well as RMS inequalities for these pickup and delivery problems (see, e.g., Hernández and Salazar-González [9]). However, a RMS inequality corresponds to a MS inequality in the symmetrized version of the problem (which is obtained by exchanging the sign of all demand numbers) and from a structural point of view, a RMS inequality is similar to a MS inequality. For this reason, we do not further explore these pickup-and-delivery variations on this paper.

The second situation, which is also motivated by a simple analysis of expression (6), is to consider variations of the CVRP where lower bound values on the arc flows are bigger than one. To illustrate such a situation, consider the so-called *Balanced Vehicle Routing Problem* (BVRP), where a minimum number of customers $\underline{Q}$ and a maximum number of customers Q are required to each route in a feasible solution. In general, the designation "balanced" applies to a variant of the problem when $Q - \underline{Q}$ is small. The BVRP is a particular version of the problem suggested and studied in [7] where besides the vehicle capacity bounds there are upper and lower bounds also for the length of the routes. A multiobjective optimization variant of this problem has been suggested and studied in [11] where the goal is to minimize the total route length as well as the difference between the longest route and the shortest route. In our paper we relax the condition that $\underline{Q}$ and Q should be close since our aim is to consider situations with a lower bound $\underline{Q}$ (> 1) and we may even allow examples where $Q = |V| - 1$ (i.e., no upper capacity on the vehicles). The lower limited capacity requirement can be easily modeled through a SCF model by setting the lower bound value $\underline{q}_a$ on the arcs leaving the depot as being equal to $\underline{Q}$. More precisely, a SCF model for the BVRP can be obtained by setting the parameters as follows:

$$\begin{aligned}
\underline{q}_a = \overline{q}_a = 0 \qquad & \text{for all } a \in \delta^-(1) \\
\underline{q}_a = \underline{Q} \text{ and } \overline{q}_a = Q \qquad & \text{for all } a \in \delta^+(1) \\
\underline{q}_a = 1 \text{ and } \overline{q}_a = Q - 1 \qquad & \text{for all } a \notin \delta^+(1) \cup \delta^-(1).
\end{aligned}$$

Thus, it is quite easy to include lower bound information in a SCF formulation. On the other hand, finding from scratch inequalities involving only the x_a variables to guarantee the minimum required number of customers in each route seems to be far from easy. Fortunately, the tool of projection applied on the SCF formulation permits us to obtain such set of inequalities. The MS inequalities are exactly the ones given in (7). Note that they do not depend on the lower bound value $\underline{Q}$. The RMS inequalities, instead, take into account the upper bound capacity as well as the new lower bound capacity:

$$\underline{Q}x(1:S) + x(S':S) \leq (Q-1)x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}. \quad (8)$$

These constraints guarantee that if a (partial) route does not have the minimum required number of customers, then it cannot be closed by connecting directly to the depot the two extremes of the route. This can easily be seen if we use the in-degree and out-degree equations (1) on the customer nodes to rewrite the RMS inequality (8) associated with a set S as follows:

$$(\underline{Q}-1)|S| \leq (Q-1)x(S:S') + (\underline{Q}-1)x(S':S) + \underline{Q}x(S:S). \qquad (9)$$

Suppose now that we have a set S where all the nodes are in the same route and connected, and that $|S| < \underline{Q}$. Then, we have that $x(S:S) = |S|-1$ and the RMS inequality becomes

$$(Q-1)x(S:S') + (\underline{Q}-1)x(S':S) \geq \underline{Q} - |S|.$$

It states that the set S must be connected to at least one node in the set S'. A more elaborate, but similar, interpretation could be found for a set S such that $|S| \geq \underline{Q}$.

The interpretation just given for the RMS inequality (8) is exactly the opposite interpretation given for the MS inequalities (7) (see, for instance, Araque et al. [1]) and that is why we have named these inequalities as *reverse multistars.*

The MS inequalities are known to define facets, under mild conditions, of the undirected polytope associated to the CVRP. Although no similar study has been done for the BVRP, we have no reason to suspect that the MS inequalities can be strengthened when non-trivial lower bounds are imposed on the vehicle capacity. In contrast, the RMS inequalities can be strengthened by decreasing the coefficient of the variables in the right hand-side term leading to inequalities that we denote by *Enhanced RMS* (ERMS) inequalities and that only involve the lower bound capacity:

$$\underline{Q}x(1:S) + x(S':S) \leq (\underline{Q}-1)x(S:S') + |S| \qquad \text{for all } S \subseteq V \setminus \{1\}. \quad (10)$$

Proposition 3.1 *The ERMS inequalities (10) are valid for the BVRP polytope.*

Proof. Consider a feasible solution x^* and a set S^*. The set S^* is partitioned into the subsets $S_1, \ldots, S_k$ where each subset S_i corresponds to a connected component of the route x^* in S^* (i.e., a path). We want to prove

$$\sum_{i=1}^{k} \underline{Q}x^*(1:S_i) + x^*(S':S_i) \leq \sum_{i=1}^{k} (\underline{Q}-1)x^*(S_i:S') + |S_i|.$$

To this end we will show that

$$\underline{Q}x^*(1:S_i) + x^*(S':S_i) \leq (\underline{Q}-1)x^*(S_i:S') + |S_i|$$

$x_{12} = 1$	$x_{13} = 1$	$x_{14} = 0.5$	$x_{16} = 0.5$	$x_{21} = 0.5$	$x_{24} = 0.5$	$x_{31} = 0.5$
$x_{36} = 0.5$	$x_{41} = 0.5$	$x_{47} = 0.5$	$x_{51} = 1$	$x_{61} = 0.5$	$x_{67} = 0.5$	

Table 1: Fractional solution violating a ERMS inequality.

for each $i = 1, \dots, k$. Indeed, let S_i be the partition subset associated to a path starting from node $j \in S'$. If $x^*_{1j} = 0$ then the claim is true because $x^*(S' : S_i) \leq |S_i|$. Otherwise, $x^*(S' : S_i) = 0$ and again the inequality holds (note that if $|S_i| < \underline{Q}$ then $x^*(S_i : S') \geq 1$). $\square$

Table 1 shows a fractional solution of a BVRP instance with $n = 7$, $Q = 3$ and $\underline{Q} = 2$. It satisfies all MS inequalities, all RMS inequalities, and all the rounded MS inequalities described in Section 4. However this fractional solution violates the ERMS inequality defined by $S = \{2, 3\}$. This suggests that the ERMS inequalities might be useful in a cutting plane approach to solve the BVRP.

The ERMS inequalities (10) were inspired by the fact that, in inequalities (9), the coefficients of the variables associated with arcs $a \in \delta^+(S) \cap \delta^-(S')$ are different from the coefficients of the variables associated with the "reverse" arcs $a \in \delta^+(S') \cap \delta^-(S)$, and one may suspect that a similar set of inequalities, with "symmetric" coefficients, might also be valid. The reason for this is that a similar set of inequalities might also be derived for the undirected version of the problem (where undirected edge variables are used instead of directed arc variables). Raising the coefficient of the variables in the cut $\delta^+(S') \cap \delta^-(S)$ from $\underline{Q} - 1$ to $Q - 1$ would lead to such a symmetric inequality that is clearly valid but does not produce the required meaning, namely guaranteeing the lower bounds. The other alternative to obtain a symmetric inequality is to decrease the coefficient of the variables in the cut $\delta^+(S) \cap \delta^-(S')$ from $Q - 1$ to $\underline{Q} - 1$. This leads to a stronger inequality which, according to the previous result, it is still valid. The stronger inequalities suggest the following question. Similarly to what happens to the weaker RMS inequalities, could we find a compact model which implies the ERMS inequalities (10)? Or in a more broad way, can they be separated in polynomial time? For the moment we do not have an answer to these two related questions. At first glance, it appears that we could obtain such a compact model by decreasing the coefficients $\overline{q}_a$ in (4) from $Q-1$ to $\underline{Q}-1$ for the arcs $a \notin \delta^+(1) \cup \delta^-(1)$ and use the projection tool as before. However, these modified upper bound constraints are not valid since they also force each route to have at most $\underline{Q}$ customers.

We end this section noting that for the special case of the BVRP where

$Q = \underline{Q}$ the ERMS inequalities (10) do not provide new information since they can be shown to be equivalent to the MS inequalities (7). We next show that the MS inequality (7) for a given set S is equivalent to the ERMS inequality (10) for the complement set S'. For the proof we use the fact that, when $Q = \underline{Q}$, all feasible solutions have a fixed number of vehicles given by $(|V|-1)/Q$, i.e. $x(1 : V \setminus \{1\}) = (|V|-1)/Q$.

Proposition 3.2 *When $Q = \underline{Q}$, the ERMS inequality (10) for set S is equivalent to the MS inequality (7) for the set S', and vice-versa.*

Proof. Consider the ERMS inequality (10) for a given set S:

$$Qx(1 : S) + x(S' : S) \leq (Q-1)x(S : S') + |S|.$$

Using the degree constraint for the depot, $x(1 : V \setminus \{1\}) = (|V|-1)/Q$, we obtain

$$|V| - 1 + x(S' : S) \leq (Q-1)x(S : S') + Qx(1 : S') + |S|.$$

Since $|V| - 1 - |S| = |S'|$ we obtain the MS inequality (7) for the set S'. □

Clearly, this equivalence no longer holds for the more general BVRP where $\underline{Q} < Q$ since we have shown that the ERMS inequalities (10) (and even the RMS inequalities (8)) are necessary for modeling the BVRP.

4 Rounded Inequalities

It is well known that "projected" inequalities can be used to produce (by adequate division and rounding) other interesting sets of inequalities. As an example, consider the MS inequalities (7) for the unit-demand CVRP. Dividing by Q these inequalities and then rounding we obtain the following *rounded MS inequalities*

$$x(1 : S) + \left\lceil \frac{Q-1}{Q} \right\rceil x(S' : S) \geq \left\lfloor \frac{1}{Q} \right\rfloor x(S : S') + \left\lceil \frac{|S|}{Q} \right\rceil$$

that correspond to

$$x(V \setminus S : S) \geq \left\lceil \frac{|S|}{Q} \right\rceil \tag{11}$$

for all $S \subseteq V \setminus \{1\}$. These inequalities are the directed version of facet-defining inequalities for the undirected CVRP polytope (see, e.g. Campos et al. [2], Cornuejols and Harche [3] and Araque et al. [1]) and are by far the most relevant inequalities in cutting-plane approaches for solving the CVRP and related problems. These constraints are a capacitated version of the well-known *subtour elimination constraints* used, for instance, in the context of the travelling salesman problem:

$$x(V \setminus S : S) \geq 1 \tag{12}$$

for all $S \subseteq V \setminus \{1\}$. When $|S| \leq \underline{Q}$ inequality (12) coincides with (11). Note also that by using the indegree constraints, inequalities (11) can be rewritten in a packing form as follows

$$x(S : S) \leq |S| - \left\lceil \frac{|S|}{Q} \right\rceil. \tag{13}$$

We show next that, by performing a similar rounding procedure starting from the ERMS inequalities (10), one can obtain a new set of inequalities that become relevant for problems with lower bound capacities. We note that a similar procedure can be applied to the weaker RMS inequalities (8). However, the obtained inequalities will be weaker than the ones obtained from the ERMS inequalities (10). For this reason we only focus on applying the derivation procedure to (10).

As it has been done with the original RMS inequalities, an ERMS inequality can be rewritten as follows:

$$(\underline{Q} - 1)|S| \leq (\underline{Q} - 1)x(S : S') + (\underline{Q} - 1)x(S' : S) + \underline{Q}x(S : S).$$

Then, if we divide this inequality by $\underline{Q}$, and then apply rounding as before, we obtain the new *rounded ERMS inequality*:

$$|S| - \left\lfloor \frac{|S|}{\underline{Q}} \right\rfloor \leq \left\lceil \frac{\underline{Q} - 1}{\underline{Q}} \right\rceil x(S : S') + \left\lceil \frac{\underline{Q} - 1}{\underline{Q}} \right\rceil x(S' : S) + x(S : S), \tag{14}$$

which is the same as

$$|S| - \left\lfloor \frac{|S|}{\underline{Q}} \right\rfloor \leq x(S : S') + x(S' : S) + x(S : S).$$

It is not difficult to see that there is no dominance relationship between the two sets of inequalities, the ERMS inequalities (10) and the rounded ERMS inequalities (14). Clearly, one expects the rounding to perform better when $|S|/\underline{Q}$ is not integer.

$x_{12} = 0.111111$	$x_{13} = 0.444444$	$x_{14} = 0.111111$	$x_{15} = 0.5$	$x_{16} = 0.5$
$x_{17} = 0.166667$	$x_{18} = 0.5$	$x_{21} = 0.777778$	$x_{28} = 0.222222$	$x_{31} = 0.111111$
$x_{32} = 0.888889$	$x_{41} = 0.777778$	$x_{48} = 0.222222$	$x_{53} = 0.055556$	$x_{54} = 0.888889$
$x_{58} = 0.055556$	$x_{65} = 0.166667$	$x_{67} = 0.833333$	$x_{71} = 0.666667$	$x_{75} = 0.333333$
$x_{83} = 0.5$	$x_{86} = 0.5$			

Table 2: Fractional solution satisfying all ERMS inequalities and violating a rounded ERMS inequality.

Table 2 presents a fractional solution for an instance with $n = 8$, $Q = 4$ and $\underline{Q} = 3$. This solution satisfies all MS inequalities (7), RMS inequalities (8), ERMS inequalities (10) and rounded MS inequalities (11). However, it is easy to see that the solution violates the rounded ERMS inequalities (14) defined by $S = \{2, 3\}$. Thus, the rounded ERMS inequalities (14) might also be useful in a cutting plane algorithm for solving instances of the BVRP. We give next some intuition on why this may happen.

First, we note that one can exhibit, quite easily, a simple case where the rounded ERMS inequality (14) is at least as strong as the corresponding (for the same set S) ERMS inequality (10). Consider $S = V \setminus \{1\}$. Then, both constraints give an upper bound on the number of vehicles leaving the depot, but the upper bound given by the rounded ERMS inequality (14) is stronger when $(|V| - 1)/\underline{Q}$ is not integer.

More generally, consider the following situation with a set S consisting of three nodes linked together in a feasible solution for the problem and where the depot (node 1) is linked to one node in S. Assume that $\underline{Q} = 5$ and $Q = 6$. The ERMS inequality (10) for these parameters becomes $5 + 0 \leq 3 + 4x(S : S')$, which is equivalent to $2/4 \leq x(S : S')$. The inequality combined with the in-degree and out-degree constraints states that the last node in this sequence must be connected to a customer (because the route has only visited three nodes). From the LP relaxation point of view we would prefer to obtain an inequality where in the same situation the left-hand side would be equal to 1. This is given precisely by the rounded ERMS inequality (14).

We can use the equality constraints of the model to obtain another useful way of expressing the rounded ERMS inequalities (14). Indeed, by using the in-degree and the out-degree constraints for every node in set S we obtain

$$2x(S : S) + x(S : 1) + x(1 : S) + x(S : S') + x(S' : S) = 2|S|,$$

and thus the rounded ERMS inequality associated with S can also be written

$x_{13} = 0.2$	$x_{14} = 1$	$x_{15} = 0.433333$	$x_{17} = 0.366667$
$x_{21} = 0.8$	$x_{25} = 0.2$	$x_{31} = 0.6$	$x_{32} = 0.4$
$x_{42} = 0.2$	$x_{43} = 0.2$	$x_{45} = 0.1$	$x_{46} = 0.5$
$x_{51} = 0.466667$	$x_{52} = 0.1$	$x_{53} = 0.3$	$x_{56} = 0.066667$
$x_{57} = 0.066667$	$x_{61} = 0.133333$	$x_{63} = 0.3$	$x_{67} = 0.566667$
$x_{72} = 0.3$	$x_{75} = 0.266667$	$x_{76} = 0.433333$	

Table 3: Fractional solution satisfying all rounded ERMS inequalities and violating an ERMS inequality.

as

$$x(S \cup \{1\} : S \cup \{1\}) \leq |S| + \left\lfloor \frac{|S|}{\underline{Q}} \right\rfloor. \tag{15}$$

Constraints (15) are subtour elimination constraints that are quite similar to constraints (13). However, here node 1 is included in the "subtour elimination" part. As far as we know, subtour elimination constraints including the depot are not common in formulations for constrained routing problems. Note also that when $|S| < \underline{Q}$ the constraints become very intuitive for this problem. Indeed, the lower bound vehicle capacity requirements on the problem can also be viewed as imposing that small subtours involving the depot are not allowed.

Still, observe that the rounded ERMS inequalities do not dominate the ERMS inequalities. Table 3 shows a fractional solution of a BVRP instance with $n = 7$, $Q = 4$ and $\underline{Q} = 3$. This solution satisfies all MS inequalities, Rounded MS inequalities and Rounded ERMS inequalities, and it violates the ERMS inequality associated with $S = \{2, 3, 4, 5\}$.

We end this section providing a similar result to the one given at the end of the previous section. For the special case of the BVRP where $Q = \underline{Q}$, the rounded ERMS inequalities (14) do not provide new information since they are equivalent to the rounded MS inequalities (11). Following the same assumptions given at the end of the previous section, we can state and prove the next result.

Proposition 4.1 *When $Q = \underline{Q}$, the rounded ERMS inequality (14) for set S is equivalent to the rounded MS inequality (11) for the set S'.*

Proof. Consider the rounded ERMS inequality (14) for a given set S. Using the in-degree equations for the nodes in set S we obtain

$$|S| - \left\lfloor \frac{|S|}{Q} \right\rfloor \leq x(S : S') + |S| - x(1 : S).$$

Using the fact that $x(1 : V \setminus \{1\}) = (|V| - 1)/Q$ and rearranging we get

$$\frac{|V|-1}{Q} - \left\lfloor \frac{|S|}{Q} \right\rfloor \leq x(S : S') + x(1 : S').$$

Since, the left-hand side of the resulting inequality is equal to $\lceil |S'|/Q \rceil$, we achieve the desired result. $\square$

As shown by the fractional solution in Table 2, the above-cited equivalence no longer holds for the more general BVRP where $\underline{Q} < Q$. However, in some cases, we obtain some interesting relations. Consider, for instance, the case when $Q = \underline{Q} + 1$. For simplicity of notation we now assume $\underline{Q} = Q - 1$. First note that in this situation we have

$$\left\lceil \frac{|V|-1}{Q} \right\rceil \leq x(1 : V \setminus \{1\}) \leq \left\lfloor \frac{|V|-1}{Q-1} \right\rfloor.$$

Consider the rounded ERMS inequality (14) for a set S

$$|S| - \left\lfloor \frac{|S|}{Q-1} \right\rfloor \leq x(S : S') + x(S' : S) + x(S : S).$$

Using the same reasoning as used in the proof of Proposition 4.1, we obtain

$$x(1 : S) + |S| - \left\lfloor \frac{|S|}{Q-1} \right\rfloor \leq x(S : S') + |S|.$$

Using $\lceil (|V| - 1)/Q \rceil \leq x(1 : V \setminus \{1\})$ and cancelling equal terms we obtain the following cut-like inequality for the set S':

$$\left\lceil \frac{|V|-1}{Q} \right\rceil - \left\lfloor \frac{|S|}{Q-1} \right\rfloor \leq x(S : S') + x(1 : S'), \tag{16}$$

which is quite similar to the rounded MS inequality (11) but with a different right-hand side. For the situations where

$$\left\lceil \frac{|V|-1}{Q} \right\rceil = x(1 : V \setminus \{1\}) = \left\lfloor \frac{|V|-1}{Q-1} \right\rfloor$$

the two sets of constraints (16) and (14) are equivalent, and then it is interesting to compare (16) and (11). We discuss next two different cases:

Case 1. $n = 8$, $Q = 4$, $\underline{Q} = 3$: We obtain $x(1 : V \setminus \{1\}) = 2$. In this case one can easily see that the inequalities (16) either are equivalent to (11) or are even weaker. More precisely, (16) is weaker than (11) if $|S'| = 1$, and the two inequalities are equivalent if $|S'| > 1$. This analysis does not invalidate the solution shown in Table 2 since in that case the solution satisfies $\lceil(|V|-1)/Q\rceil < 2.333 = x(1 : V \setminus \{1\})$. However, if we had previously included the degree constraint $x(1 : V \setminus \{1\}) \leq \lfloor(|V|-1)/(\underline{Q}-1)\rfloor = 2$ (which is equivalent to a rounded ERMS (14) for $S = V \setminus \{1\}$) we would be in the situation of applying the analysis given here in Case 1.

Case 2. $n = 10$, $Q = 4$, $\underline{Q} = 3$: Here we have $x(1 : V \setminus \{1\}) = 3$. It is interesting to point that for this case, in three out of the nine possibilities, inequality (16) is stronger than the corresponding (for the same set) rounded MS inequality (11), and they are equivalent for the remaining possibilities. Thus, for this case, the rounded ERMS inequalities (14) make the rounded MS inequalities (11) redundant. For instance, when $|S'| = 4$ the rounded MS inequality (11) has a right-hand side equal to 1, while inequality (16) for S with $|S| = 5$ (and thus, $|S'| = 4$ making the constraint comparable with (11)) has the right-hand side equal to 2, thus it is stronger. There is an interesting interpretation to this: consider a set S' such that $|S'| = 4$; the rounded MS inequality (11) states that at least one arc incoming into S' has positive value for the corresponding variables x_a; however if there is only one arc in this situation then all the nodes in S' must be included in one vehicle, which will be full; but then, the lower bound capacity makes it impossible to put all the remaining 5 nodes into feasible routes. In other words, at least two arcs must enter S' with positive value for the corresponding variables x_a, as stated by the corresponding rounded ERMS inequality (14). A similar analysis holds for sets S' with $|S'| = 7$ and 8.

5 CVRP with a fixed number of vehicles

In the previous section, we have studied a problem where the lower bound on the number of customers per route was part of the problem specification. The main reason for the proposed study was to motivate a problem where RMS inequalities are important for modeling the problem. In this section we consider a standard variant of the CVRP where the problem specification does not explicitly include lower bound information. However, sometimes this lower bound is implicit on the problem specification, and thus it might be used to generate new valid inequalities for the problem.

Consider the CVRP with unit demands, an upper bound Q on the number of customers per route, and a fixed number m of vehicles given explicitly

$x_{12} = 0.25$	$x_{13} = 0.275$	$x_{14} = 0.4$	$x_{15} = 0.325$	$x_{16} = 0.25$	$x_{17} = 0.25$
$x_{18} = 0.25$	$x_{21} = 0.75$	$x_{26} = 0.125$	$x_{28} = 0.125$	$x_{31} = 0.275$	$x_{32} = 0.175$
$x_{35} = 0.55$	$x_{41} = 0.4$	$x_{42} = 0.2$	$x_{43} = 0.4$	$x_{51} = 0.575$	$x_{54} = 0.3$
$x_{57} = 0.125$	$x_{62} = 0.375$	$x_{67} = 0.625$	$x_{73} = 0.325$	$x_{75} = 0.05$	$x_{78} = 0.625$
$x_{84} = 0.3$	$x_{85} = 0.075$	$x_{86} = 0.625$			

Table 4: Fractional solution violating an ERMS inequality.

by the equation

$$x(1 : V \setminus \{1\}) = x(V \setminus \{1\} : 1) = m. \tag{17}$$

Most of the exposition given in this section also applies to the more realistic setting where an upper bound on the number of vehicles is given. In order to show how a lower bound capacity can be useful for such a problem, let us consider an instance with $n = 8$, $Q = 4$ and $m = 2$. It is clear that in any feasible solution for the problem, each vehicle cannot visit less than three customers. Thus, we can reformulate the problem and add this lower bound information on the capacity of the vehicles. More generally, this CVRP variant is a BVRP with $\underline{Q} = (|V| - 1) - (m - 1)Q$ and a degree constraint on the depot.

The question now is to know whether, without being strictly necessary to write a valid formulation for the problem, the lower bound information is useful for developing inequalities that will tighten the LP relaxations of natural formulations for the problem with a fixed number of vehicles (or an upper bound on the number of vehicles).

Table 4 shows a fractional solution for an instance with $n = 8$, $Q = 4$ and $m = 2$. As noted before, $\underline{Q} = 3$. This solution satisfies the MS inequalities (7), rounded MS inequalities (11) and the degree constraint (17) stating that m vehicles must be used. However, it violates an ERMS inequality (10) for the set $S = \{2, 3, 4, 5\}$.

This solution tells us that the ERMS inequalities (10) might be useful in the context of a pure cutting plane method using only the x_a variables. We now show that the same does not hold for the rounded ERMS inequalities (14).

Proposition 5.1 *For the CVRP with a fixed number of vehicles m, the rounded ERMS inequality for set S' is implied by the rounded MS inequality for set S.*

Proof. For the CVRP with a fixed number of vehicles m, we have $\underline{Q} =$

$(n-1)-(m-1)Q$. Let us assume that $\underline{Q} = Q - q$ for a certain integer number q. Since $Qm = n-1+q$, the number $(n-1+q)/Q$ is integer.

Using arguments shown before, the rounded ERMS inequalities for set S can be rewritten as:

$$\frac{n-1+q}{Q} + \left\lfloor \frac{|S|}{Q-q} \right\rfloor \leq x(S:S') + x(1:S'),$$

which is quite similar to the rounded MS inequality for the set S':

$$\left\lceil \frac{|S'|}{Q} \right\rceil \leq x(S:S') + x(1:S').$$

Thus we only need to compare the left-hand sides of the two inequalities.

Without loss of generality, assume that $|S| = kQ + p$ and $|S'| = k'Q + p'$ with $0 \leq p \leq Q-1$ and $0 \leq p' \leq Q-1$. Note that $k + k' = m - 1$ and $p + p' = \underline{Q}$.

We first consider that case where $p' > 0$. Under this assumption, $\lceil |S'|/Q \rceil = k'+1$ and $\lfloor |S|/(Q-q) \rfloor \geq k'$ since $|S| = k(Q-q)+p+kq$. Thus

$$\left\lceil \frac{|S'|}{Q} \right\rceil = k' + 1 = m - k \geq \frac{n-1+q}{Q} - \left\lfloor \frac{|S|}{Q-q} \right\rfloor$$

and therefore the rounded MS inequality for set S' dominates the rounded ERMS inequality for set S.

Consider now the case where $p' = 0$. Then $|S| = kQ + Q - q$ and $|S'| = k'Q$. Under this assumption, $\lceil |S'|/Q \rceil = k'$ and $\lfloor |S|/(Q-q) \rfloor \geq k'+1$ since $|S| = k(Q-q)+(Q-q)+kq$. We have obtained the desired inequality. □

This result is a bit surprising considering the examples for the case $\underline{Q} = Q - 1$ given in the previous section. However this can be explained because, for this special version of the CVRP, the values of Q and m identify univocally one value for $\underline{Q}$. An example of this dominance is Case 1 given at the end of Section 4. On the other hand, in the BVRP we may have other situations with fixed values of Q and $\underline{Q}$, and several feasible values for m.

The fact that the ERMS inequalities (10) may be of interest for this special case of the CVRP raises several possibilities, namely that a more through study of the BVRP polytope may lead to other inequalities of interest for this variant of the CVRP. A related question is to know whether the ERMS inequalities are really new for the CVRP with a degree constraint on the depot. That is, could they be shown to be equivalent to other inequalities which are already known from the literature? This is a difficult question to answer since there are many classes of inequalities for the CVRP. The best we can say is that from this large class of inequalities, and as far as we know, only the degree constraints use information on the number of vehicles, and thus it is highly unlikely that the ERMS inequalities are equivalent to other inequalities.

6 Computational results

In this section we present computational results to evaluate our contributions for solving BVRP instances. We show that using the new inequalities one can solve larger instances. To this end we compare three approaches: an ILP solver on a compact SCF model given in Section 3, and two branch-and-cut implementations. The first branch-and-cut implementation uses the standard inequalities for the capacitated VRP and only the RMS constraints (8). The second branch-and-cut implementation uses the standard inequalities for the CVRP as in the first implementation, but uses the ERMS (10) and the rounded ERMS (15) instead of the RMS constraints. The standard inequalities for the CVRP that we use in both implementations are the subtour elimination constraints (12) and the rounded MS inequalities (11). These inequalities are separated through an integrated procedure. We describe next several steps detailing this procedure. Each step is repeated while it generates violated inequalities before proceeding to the next step.

Step 1: Find violated subtour elimination constraints (12) as it is standard in the literature (see, e.g., [14]).

Step 2: Find violated rounded ERMS inequalities (15). For separating these constraints we use a heuristic approach. It consists of finding a most violated weaker version of inequality (15). This weaker inequality is

$$x(S \cup \{1\} : S \cup \{1\}) \leq |S| + \frac{|S|}{\underline{Q}},$$

which can also be rewritten in a cut-form as

$$x(S \cup \{1\} : S') + \frac{|S|}{\underline{Q}} \geq x(1 : V \setminus \{1\}).$$

These inequalities can be exactly separated by using a min-cut algorithm on the following capacitated directed network. Let x^* be the given fractional solution. Let the node set be V and a dummy node

$n+1$. Let the arc set be the arcs in A and also a dummy arc $(i, n+1)$ for each $i \in V \setminus \{1\}$. The capacity of each arc $a \in A$ is x^*_a and the capacity of each arc $(i, n+1)$ is $1/\underline{Q}$. If the capacity of an optimal min-cut separating 1 from $n+1$ in this network is smaller than $x^*(1 : V \setminus \{1\})$ then we have a set S^* defining a violated inequality (15). Otherwise, we still check this inequality for a potential violation. The violated rounded ERMS inequalities found are added to the model.

Step 3: Find violated rounded MS inequalities (11). As in Step 2 we apply a heuristic procedure, based on finding a most violated inequality of type $x(V \setminus S : S) \geq |S|/Q$, which is a weaker version of the rounded MS inequality. Again, we use a min-cut algorithm in a network similar to the one described in Step 2. The difference is that the capacity of an arc $(i, n+1)$ is now $1/Q$, and the threshold to compare the capacity of the min-cut is now $|V|/Q$. Each violated rounded MS inequality is added to the model.

Step 4: Solve the linear system (3)–(4) with $\underline{q}_a$ and $\overline{q}_a$ given in Section 3. If this system is infeasible, the dual extreme ray divides the node set $V \setminus \{1\}$ in two sets, S and S'. The ERMS inequalities (10) associated with S and S' are checked for potential violation. If one such inequality is violated then it is added to the model and the step is repeated. If no ERMS inequality is violated then we check for violation the rounded ERMS inequalities (15) associated to the same sets S and S'. If one such inequality is violated then it is added to the model and the step is repeated. When testing the approach that only uses RMS inequalities (that is, the first branch-and-cut implementation) we also use this step. However we check the RMS inequalities (8) associated with the sets S and S' instead of the ERMS and rounded ERMS inequalities.

In the description of Step 4, we could have included the separation of the MS inequalities (7) in a similar way as it was done for the RMS inequalities (8). However, our computational experiments indicated a worse performance when the MS separation was included. We do not have an explanation for this behaviour, but it coincides with an observation in Hall [8] on a related problem. We think that in our implementation the effect of the MS inequalities are nullified by the inclusion of the rounded MS inequalities. Indeed, we have observed that by including both families of inequalities the model becomes bigger, which increases the computational time without increasing the LP relaxation bounds.

Step 4 checks the feasibility or infeasibility of the linear system (3)–(4). For the arguments given in Section 2 Step 4 is an exact separation procedure for the RMS inequalities in our first branch-and-cut implementation. Unfortunately we do not have exact separation algorithms for ERMS and rounded ERMS inequalities to be used in our second branch-and-cut implementation. However, based on our experiments, the same procedure is a successful heuristic separation procedure to find violated ERMS and rounded ERMS inequalities.

In our computational experiments we have used 46 BVRP instances that were generated from two well-known CVRP instances taken from the VRP-library: the EIL101 instance with 100 customers and Euclidean distances, and the A071-03f instance with 70 customers and asymmetric distances. From each CVRP instance, we have created a family of 23 BVRP instances as follows. Each BVRP contains all the customers and the same distance matrix as the CVRP instance. The demand of all customers are equal to 1, and we have considered 3 different values of Q and several values of $\underline{Q}$. We have considered only unit-demand BVRP instances to better understand the impact of considering the lower bound $\underline{Q}$. The algorithm was coded in C++ on a Linux platform running in a Intel(R) Core(TM)2 CPU 6700 @ 2.66GHz desktop computer with 2GB RAM. We have used the CPLEX 12.1 callable library as a framework to solve the compact model and to implement the two branch-and-cut approaches. We have used the default primal heuristic and branching strategy available in this library.

We first make a quick observation on using the compact SCF model as specified in Section 3. For all the instances tested in our computational experiment, except one, the use of the model within the CPLEX package did not lead to the determination of the optimal solution within two hours. The exception was the BVRP instance based on A071-03f with $Q = 26$ and $\underline{Q} = 1$ (that is, a standard CVRP instance), which was solved to optimality in 6704 seconds. Thus, from now on, we will only focus on comparing the two branch-and-cut approaches.

Table 5 shows the results obtained with the second implementation, i.e., the one that uses all the new inequalities proposed in this paper. Table 6 shows the results obtained with the first implementation, i.e., the one which does not use the new inequalities. Both tables give information at the root node and at the end of the branch-decision search tree. The numbers of generated inequalities are in columns with labels starting with #. These columns give the numbers of inequalities at the end of the search tree if the labels end with ', and give the numbers of inequalities at the end of the root node otherwise. #SEC gives the number of subtour elimination inequalities (12), #CAP gives the number of rounded MS inequalities (11), #RMS gives the number of RMS inequalities (8), #ERMS gives the number of ERMS inequalities (10), and #RERMS gives the number of rounded ERMS inequalities (15). From left to right, other labels are the following: designation of the instance, vehicle capacities Q and $\underline{Q}$, objective value *r-LB* at the end of the root node, computing time in seconds *r-time* at the end of the root node, lower bound on the optimal objective value *LB-opt* at the end of the search, upper bound on the optimal objective value *UB-opt* at the end of the search, total computing time *tot-time* in seconds, time *sep-time* consumed by the separation procedures, number *nodes* of branch-and-bound nodes, number m of routes in the obtained optimal solution, maximum number *max* of customers in a route in the optimal solution, and minimum number *min* of customers in a route in the optimal solution. When an instance could not be solved within the time limit, we indicate this by writing "2 hours" in column *tot-time* and by writing the best heuristic objective value in column *UB-opt*.

Comparing Tables 5 and 6 we can see that bigger computational times are needed to solve the same instances with the first implementation, i.e. when using the RMS inequalities instead of the ERMS and rounded ERMS inequalities. These results show that the new inequalities are worth having in a cutting plane method for solving the BVRP. The results also indicate that, in general, the effect of having the new inequalities is stronger for the asymmetric instances and also when the upper capacity value becomes smaller. In terms of the best approach, the results (see Table 5) indicate that the Euclidean instances are easier to solve than the asymmetric instances. Over the 46 BVRP instances, four instances are not solved to optimality within the time limit by the second implementation (Table 5) while this number goes to fifteen instances when using the first implementation (Table 6). Moreover, the first branch-and-cut implementation finished with no feasible BVRP solution in four over these fifteen instances (and Columns *UB-opt*, m, *max* and *min* do not contain numbers in the table). Recall that the primal-heuristic approach in both branch-and-cut implementations is the default approach in CPLEX.

An interesting observation regards the columns *r-LB* in Tables 5 and 6. The results show that the inclusion of the RMS inequalities in the first implementation (see Table 6) do not alter by much the LP bound given by the instance when $\underline{Q} = 1$. In fact, the LP bound increases very slowly (when it increases) with the value of $\underline{Q}$, for a fixed value of Q. This can be explained by analyzing the solutions of the LP relaxation of the SCF model. In most of the cases, the LP solution contains large fractional routes, implying that the upper bound inequalities on f_a in (4) are either satisfied at equality or close to being satisfied at equality. This means that adding the lower bound inequalities on f_a in (4) may not have any effect on the given LP bound if the given $\underline{Q}$ is not close to the given Q. The LP bounds improve when the ERMS and rounded ERMS inequalities are added. Although this improvement is not always considerable at the root node of the search tree, it should be noted that a characteristic of BVRP is that the lower bound information appears to be more effective on "sparse" graphs (that is, on instances where the number of arcs is much smaller than the maximum allowed number of arcs). For that reason the effect of the inequalities become stronger in deeper nodes of the search tree, when variables associated to arcs of the graph start being fixed either to 1 or to 0.

Finally, we look at the effect of using the ERMS inequalities in a standard cutting plane method for the classic CVRP with a fixed number m of vehicles that must be used (Section 5). As before we solve each instance with two branch-and-cut implementations. One implementation only separates subtour elimination constraints (12) and the rounded MS inequalities (11), thus it solves CVRP with a degree constraint on the depot. The other implementation also uses the ERMS and rounded ERMS inequalities based on the fact that, since m is fixed, one has a lower bound vehicle capacity $\underline{Q} = n-(m-1)Q$. Notice that, as we have proved in Section 5, the rounded ERMS inequalities are redundant in the presence of the rounded MS inequalities for this specific variation. However, since we are using a heuristic separation procedure for the rounded MS inequalities, some rounded ERMS inequalities may be useful and we decided to have both separation routines in the second implementation. Tables 7 and 8 report these results. By analyzing only the computational times we can conclude that an approach does not dominate the other. Indeed, in some cases we see better solution times when the new inequalities are included and in some cases we see better solution times when they are not. Observing the r-LB columns in both tables, in most instances the LP bound at the root node is higher when the new inequalities are separated, but not always. This is explained by the use of heuristic separations for some families of inequalities. It is still possible that more sophisticated separation procedures could lead to more clear advantages when using the new inequalities.

7 Conclusions

In this paper we have examined a subset of the inequalities obtained by projecting the flow variables of a standard single flow formulation for the CVRP. These are the reversed multistar inequalities (RMS). For the CVRP these inequalities were observed to be redundant. However, we have shown that for other variants of the CVRP (namely the CVRP with lower bound capacities or BVRP) they become rather interesting.

In the context of the BVRP, the new set of inequalities have also suggested two other families of inequalities, namely the Enhanced RMS (ERMS) inequalities and the rounded ERMS inequalities. We have also developed a branch-and-cut method for the BVRP which uses the inequalities proposed in this paper. Computational results indicate that the proposed approach solves reasonably well BVRP instances from the VRP library, with up to 100 nodes. We have also shown the relevance of using the ERMS and rounded ERMS inequalities in contrast to using only the original RMS inequalities.

We have also pointed out a relation between lower vehicle capacity information with the CVRP with a fixed number of vehicles. However, it is not clear from our experiments whether this relation can be effectively used in practice.

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name	Q	$\underline{Q}$	#SEC	#CAP	#ERMS	#RERMS	r-LB	r-time	#SEC'	#CAP'	#ERMS'	#RERMS'	LB-opt	UB-opt	tot-time	sep-time	nodes	m	max	min
eilA101	38	1	72	58	0	0	650.69	5.2	188	118	0	0	655.00	655	27.8	19.6	716	3	38	28
eilA101	38	28	71	60	69	15	652.22	5.5	199	113	152	26	655.00	655	34.9	25.3	810	3	38	28
eilA101	38	29	79	68	82	19	652.77	7.2	251	175	269	59	657.00	657	106.6	65.9	2471	3	37	29
eilA101	38	30	64	58	83	23	652.85	5.8	205	141	233	58	657.00	657	73.4	45.6	1842	3	36	30
eilA101	38	31	68	58	88	25	653.42	6.4	499	256	622	182	660.00	660	430.5	213.6	10535	3	38	31
eilA101	38	32	75	72	117	41	653.00	7.9	1077	584	2366	854	662.50	664	2 hours	1732.9	93028	3	36	32
eilA101	28	1	74	71	0	0	672.44	5.1	82	72	0	0	674.00	674	6.5	2.2	11	4	28	20
eilA101	28	20	55	52	57	14	671.50	3.9	64	53	57	14	674.00	674	5.2	1.8	21	4	28	20
eilA101	28	21	58	51	63	17	672.47	4.5	94	59	80	21	676.00	676	12.8	7.3	447	4	28	21
eilA101	28	22	53	50	69	21	671.98	4.2	113	66	117	34	676.00	676	15.9	8.8	410	4	27	22
eilA101	28	23	61	51	74	25	672.50	5.1	221	164	392	149	678.00	678	84.4	48.5	2778	4	27	23
eilA101	28	24	59	54	83	31	672.62	5.6	338	281	816	337	680.00	680	346.1	131.5	8671	4	27	24
eilA101	28	25	69	65	114	43	675.08	7.7	608	521	2255	976	684.00	684	3430.1	496.7	29668	4	25	25
eilA101	23	1	72	76	0	0	695.02	5.6	405	688	0	0	704.00	704	273.6	134.6	7012	5	23	13
eilA101	23	12	68	57	61	11	697.00	4.4	255	258	288	69	704.00	704	142.3	78.5	5202	5	22	14
eilA101	23	13	68	71	93	19	697.73	7.0	227	209	270	61	704.00	704	100.3	63.5	3312	5	23	13
eilA101	23	14	66	52	64	14	697.31	4.8	299	258	415	110	704.00	704	197.7	102.9	5014	5	22	14
eilA101	23	15	61	54	72	16	697.40	5.1	416	400	647	201	705.00	705	284.8	114.0	5519	5	23	16
eilA101	23	16	70	53	73	17	698.48	5.2	343	253	469	124	705.00	705	222.4	101.3	5527	5	23	16
eilA101	23	17	48	42	69	26	698.50	4.2	321	279	698	248	706.00	706	245.5	100.7	5750	5	22	17
eilA101	23	18	53	45	80	30	699.50	4.1	281	142	474	173	707.00	707	188.3	119.4	8213	5	22	18
eilA101	23	19	58	48	85	33	700.65	5.3	571	338	1567	623	708.00	708	1323.3	350.3	19056	5	22	19
eilA101	23	20	58	54	99	41	701.75	6.0	907	326	3081	1274	707.38	721	2 hours	1004.9	45010	5	20	20
A071-03f	26	1	26	74	0	0	2030.00	1.3	100	954	0	0	2092.00	2092	45.5	22.1	1940	3	26	18
A071-03f	26	18	18	71	101	30	2035.70	1.7	78	585	650	146	2092.00	2092	49.3	23.0	2260	3	26	18
A071-03f	26	19	22	74	95	28	2037.43	1.9	48	311	441	119	2092.00	2092	17.4	9.1	785	3	26	19
A071-03f	26	20	21	54	81	22	2041.00	1.7	88	477	764	228	2105.00	2105	63.5	27.0	3280	3	24	22
A071-03f	26	21	17	32	52	17	2037.50	1.0	77	527	1065	370	2105.00	2105	76.7	25.4	2728	3	25	21
A071-03f	26	22	28	64	101	35	2037.50	1.5	123	768	1785	648	2105.00	2105	183.0	34.8	3083	3	24	22
A071-03f	26	23	27	73	141	63	2037.20	2.4	143	911	2672	1136	2109.00	2109	400.6	62.0	5839	3	24	23
A071-03f	20	1	25	71	0	0	2142.50	1.3	201	4291	0	0	2246.00	2246	1000.1	193.8	23314	4	20	11
A071-03f	20	11	29	89	93	12	2153.50	2.0	221	3113	1006	107	2246.00	2246	744.1	134.0	15456	4	20	11
A071-03f	20	12	19	38	46	11	2153.75	1.0	166	2537	1104	154	2253.00	2253	777.1	166.5	20564	4	20	13
A071-03f	20	13	22	49	59	12	2165.46	1.3	160	2302	1587	254	2253.00	2253	667.0	128.4	15023	4	20	13
A071-03f	20	14	25	65	110	40	2157.25	1.7	157	2143	1989	395	2257.00	2257	931.6	166.5	20440	4	19	14
A071-03f	20	15	26	68	97	28	2169.00	1.8	227	1984	2841	668	2258.00	2258	1068.4	124.2	13216	4	20	15
A071-03f	20	16	23	74	113	37	2175.88	1.9	214	1981	5636	1657	2262.50	2272	2 hours	607.4	75823	4	20	16
A071-03f	20	17	17	36	68	31	2177.20	1.4	202	1242	6112	2119	2283.00	2283	6608.3	369.3	50408	4	18	17
A071-03f	16	1	22	97	0	0	2291.78	2.3	213	5306	0	0	2406.00	2406	1371.4	143.3	12059	5	16	8
A071-03f	16	8	24	66	59	7	2296.50	1.8	201	4434	575	67	2406.00	2406	869.3	133.7	12708	5	16	8
A071-03f	16	9	20	54	58	7	2284.50	1.2	154	3391	1092	108	2406.00	2406	715.3	113.5	12691	5	16	9
A071-03f	16	10	16	59	74	16	2289.67	1.6	141	1924	899	119	2406.00	2406	306.6	72.2	8316	5	16	10
A071-03f	16	11	17	73	95	23	2298.38	1.8	132	1942	1441	200	2416.00	2416	624.6	113.9	15377	5	16	11
A071-03f	16	12	23	86	129	41	2306.00	2.2	246	3303	4105	767	2416.00	2416	2789.4	192.1	17524	5	16	12
A071-03f	16	13	26	57	98	35	2323.17	2.0	212	2024	4506	1099	2443.00	2443	4205.8	242.8	31057	5	16	13
A071-03f	16	14	17	31	76	35	2335.27	1.9	302	1041	6838	2879	2441.25	2485	2 hours	195.6	14603	5	14	14

Table 5: BVRP (results with CVRP, ERMS and rounded ERMS inequalities).

name	Q	$\underline{Q}$	#SEC	#CAP	#RMS	r-LB	r-time	#SEC'	#CAP'	#RMS'	LB-opt	UB-opt	tot-time	sep-time	nodes	m	max	min
eilA101	38	1	72	58	0	650.69	5.0	188	118	0	655.00	655	27.2	19.0	716	3	38	28
eilA101	38	28	70	58	12	651.83	5.4	266	183	63	655.00	655	61.1	41.6	1396	3	38	28
eilA101	38	29	67	56	20	651.96	5.0	465	384	189	657.00	657	367.3	169.4	7453	3	36	30
eilA101	38	30	65	56	19	650.90	4.8	772	585	342	657.00	657	759.7	415.3	18135	3	36	30
eilA101	38	31	56	50	26	650.59	4.6	535	371	458	660.00	660	873.4	591.3	36597	3	36	31
eilA101	38	32	66	51	22	651.03	5.1	1659	1001	1795	656.63	665	2 hours	2536.8	109101	3	34	32
eilA101	28	1	74	71	0	672.44	5.1	82	72	0	674.00	674	6.5	2.1	11	4	28	20
eilA101	28	20	62	55	14	671.32	5.0	72	57	14	674.00	674	6.5	2.2	22	4	28	20
eilA101	28	21	60	56	22	672.04	4.9	241	141	88	676.00	676	57.3	29.3	1320	4	28	21
eilA101	28	22	56	48	14	671.22	4.3	319	219	181	676.00	676	111.6	55.8	2424	4	27	22
eilA101	28	23	63	52	24	669.82	4.5	400	184	331	678.00	678	373.7	263.1	20201	4	27	23
eilA101	28	24	55	48	20	669.80	4.2	687	313	1173	680.00	680	2800.1	1273.8	111249	4	27	24
eilA101	28	25	51	37	28	671.06	4.0	1630	767	3584	673.52	—-	2 hours	733.0	30370	-	-	-
eilA101	23	1	72	76	0	695.02	5.5	405	688	0	704.00	704	270.3	127.9	7012	5	23	13
eilA101	23	12	66	57	15	695.00	5.1	273	470	95	704.00	704	288.5	200.2	13988	5	22	14
eilA101	23	13	63	65	16	694.26	4.5	648	1114	227	704.00	704	885.9	290.8	14795	5	23	13
eilA101	23	14	58	61	19	694.17	4.7	802	1364	397	704.00	704	1342.6	480.0	24690	5	22	14
eilA101	23	15	58	63	17	694.27	5.3	733	1359	544	705.00	705	1623.0	607.4	33343	5	23	16
eilA101	23	16	69	71	24	694.65	5.6	661	1239	581	705.00	705	1072.7	378.6	22298	5	22	16
eilA101	23	17	58	60	22	690.89	4.3	1153	1962	1717	706.00	706	5640.3	1260.7	76522	5	22	17
eilA101	23	18	61	58	30	694.75	4.9	1392	2115	3104	698.74	736	2 hours	851.8	36416	5	22	18
eilA101	23	19	60	48	31	689.11	5.0	1102	1407	4166	697.12	730	2 hours	433.9	17001	5	21	19
eilA101	23	20	55	47	46	688.37	4.6	997	595	5245	693.84	—-	2 hours	393.9	15148	-	-	-
A071-03f	26	1	26	74	0	2030.00	3.2	100	954	0	2092.00	2092	46.9	21.71	1940	3	26	18
A071-03f	26	18	21	52	13	2031.00	1.5	69	612	139	2092.00	2092	29.8	16.29	1902	3	26	18
A071-03f	26	19	23	37	13	2021.61	1.2	136	959	312	2092.00	2092	186.5	83.49	7162	3	26	19
A071-03f	26	20	18	33	12	2028.50	3.4	146	980	427	2105.00	2105	231.7	97.9	12440	3	25	21
A071-03f	26	21	18	32	11	2035.00	1.0	174	1323	840	2105.00	2105	552.9	181.82	21854	3	25	21
A071-03f	26	22	19	42	21	2029.57	1.6	234	1837	1859	2105.00	2105	1001.6	195.44	24520	3	24	22
A071-03f	26	23	20	25	15	2032.70	1.1	494	2616	5653	2099.00	2109	2 hours	533.33	53265	3	24	23
A071-03f	20	1	25	71	0	2142.50	1.3	201	4291	0	2246.00	2246	949.8	181.18	23314	4	20	11
A071-03f	20	11	19	27	7	2148.00	0.8	216	4092	263	2246.00	2246	1100.2	169.88	21168	4	20	11
A071-03f	20	12	21	41	10	2160.00	1.3	282	5600	412	2253.00	2253	2522.1	304.28	38189	4	20	12
A071-03f	20	13	17	26	12	2142.50	0.8	325	7976	775	2242.75	2263	2 hours	653.04	71754	4	20	13
A071-03f	20	14	19	24	9	2162.00	0.9	383	8698	1269	2242.00	2260	2 hours	509.51	51657	4	20	14
A071-03f	20	15	15	43	24	2144.50	1.2	418	8865	1569	2232.21	2311	2 hours	464.77	39201	4	20	15
A071-03f	20	16	15	27	17	2128.66	1.1	510	5639	3247	2224.00	2281	2 hours	480.79	43626	4	20	16
A071-03f	20	17	17	37	30	2162.80	1.3	444	2804	5839	2205.25	2378	2 hours	283.78	22301	4	18	17
A071-03f	16	1	22	97	0	2291.78	2.1	213	5306	0	2406.00	2406	1262.9	129.39	12059	5	16	8
A071-03f	16	8	25	76	12	2281.12	1.7	287	6587	277	2406.00	2406	3379.3	215.82	21336	5	16	10
A071-03f	16	9	20	63	8	2276.50	1.4	211	4308	278	2406.00	2406	1389.5	125.22	12585	5	16	10
A071-03f	16	10	19	42	9	2267.53	1.4	284	7983	700	2406.00	2406	6070.3	310.03	30551	5	16	10
A071-03f	16	11	24	46	20	2293.52	1.1	244	4670	545	2416.00	2416	2024.7	234.82	31929	5	16	11
A071-03f	16	12	18	33	17	2268.07	1.2	465	7320	2354	2357.08	2452	2 hours	318.29	24193	5	16	12
A071-03f	16	13	19	42	22	2277.68	1.8	540	4857	4943	2330.84	–	2 hours	260.01	18396	–	–	–
A071-03f	16	14	14	31	28	2269.27	1.4	439	1327	7471	2330.45	–	2 hours	207	13492	–	–	–

Table 6: BVRP (results with CVRP and RMS inequalities).

Name	Q	$\underline{Q}$	m	#SEC	#CAP	#ERMS	#RERMS	r-LB	r-time	#SEC'	#CAP'	#ERMS'	#RERMS'	LB-opt	UB-opt	tot-time	sep-time	nodes	max	min
eilA101	55	45	2	75	59	70	15	637.75	6.9	105	66	79	16	640.00	640	11.4	5.2	110	52	48
eilA101	54	46	2	79	65	84	20	636.83	6.5	190	122	164	31	640.00	640	37.2	19.1	459	52	48
eilA101	53	47	2	78	61	71	12	637.17	6.4	111	68	85	17	640.00	640	14.5	6.3	133	52	48
eilA101	52	48	2	89	73	90	20	637.42	8.2	146	96	125	29	640.00	640	19.4	8.3	181	52	48
eilA101	51	49	2	79	56	74	18	637.42	7.1	408	276	386	91	643.00	643	192.1	113.2	4304	51	49
eilA101	38	24	3	69	57	59	10	652.07	4.9	264	246	176	31	655.00	655	102.0	49.3	1698	38	28
eilA101	37	26	3	62	59	66	13	652.32	5.0	379	380	366	68	657.00	657	213.3	116.2	4760	36	30
eilA101	36	28	3	72	84	86	16	652.50	6.4	242	186	210	38	657.00	657	71.0	44.6	1770	36	30
eilA101	35	30	3	82	76	94	18	653.41	6.5	338	319	417	80	661.00	661	420.2	272.8	17675	35	30
eilA101	34	32	3	66	68	94	28	652.70	7.2	1312	1726	2523	729	659.75	665	2 hours	1710.3	73282	34	32
eilA101	28	16	4	62	58	57	10	671.56	4.4	66	61	58	10	674.00	674	6.0	2.0	27	28	20
eilA101	27	19	4	70	64	75	18	672.17	4.9	141	115	126	31	676.00	676	22.7	12.6	582	27	22
eilA101	26	22	4	70	70	91	24	674.90	6.4	501	728	883	199	682.00	682	690.1	228.3	12078	26	24
eilA101	22	11	5	68	65	62	10	699.43	5.1	206	247	182	30	704.00	704	74.9	41.8	1877	22	14
eilA101	21	16	5	67	70	93	23	700.25	5.6	650	1036	952	223	712.00	712	5519.4	1844.0	135650	21	16
eilA101	18	10	6	58	51	68	16	725.82	4.4	651	2064	684	103	739.00	739	2257.8	671.7	38779	18	11
eilA101	17	15	6	69	68	105	35	729.63	5.5	701	1918	1683	439	741.79	744	2 hours	1491.8	97322	17	15
A071-03f	37	33	2	25	37	42	5	1969.00	1.0	26	37	42	5	1979.00	1979	1.1	0.5	12	37	33
A071-03f	36	34	2	21	23	29	6	1968.00	0.7	25	25	32	6	1985.00	1985	0.9	0.4	13	35	35
A071-03f	35	35	2	19	26	39	13	1954.00	1.0	30	44	69	25	1985.00	1985	1.7	0.7	34	35	35
A071-03f	26	18	3	34	77	96	20	2035.00	1.6	105	729	848	185	2092.00	2092	74.8	30.7	2983	26	18
A071-03f	25	20	3	23	75	98	26	2037.50	1.6	138	738	858	166	2105.00	2105	110.2	40.1	4666	25	21
A071-03f	24	22	3	24	48	69	21	2036.20	1.3	113	735	935	209	2105.00	2105	107.1	32.5	3410	24	22
A071-03f	20	10	4	22	63	41	6	2138.00	1.3	200	3902	717	75	2246.00	2246	977.3	169.0	19811	20	11
A071-03f	19	13	4	21	77	100	26	2180.25	1.7	298	3979	2249	318	2257.00	2257	2449.8	236.2	24176	19	13
A071-03f	18	16	4	20	54	79	25	2174.75	1.5	291	4165	3769	568	2281.00	2281	4553.9	326.1	38555	18	16
A071-03f	16	6	5	20	74	58	3	2278.75	1.4	192	3786	322	30	2406.00	2406	833.5	125.4	12577	16	8
A071-03f	15	10	5	24	47	60	16	2301.77	1.2	347	6950	2156	239	2439.00	2439	6553.0	437.8	51638	15	10

Table 7: CVRP with a fix number of vehicle (results with CVRP, ERMS and rounded ERMS inequalities).

Name	Q	$\underline{Q}$	m	#SEC	#CAP	r-LB	r-time	#SEC'	#CAP'	LB-opt	UB-opt	tot-time	sep-time	nodes	max	min
eilA101	55	45	2	85	72	636.94	5.6	125	80	640.00	640	11.4	5.9	169	52	48
eilA101	54	46	2	82	69	637.17	5.8	145	97	640.00	640	13.4	6.9	206	52	48
eilA101	53	47	2	76	66	637.17	5.3	109	75	640.00	640	10.5	5.4	150	52	48
eilA101	52	48	2	88	75	636.97	5.5	122	88	640.00	640	12.2	6.7	192	52	48
eilA101	51	49	2	85	74	637.17	5.2	403	270	643.00	643	120.2	90.5	3873	51	49
eilA101	38	24	3	84	75	652.02	5.2	210	181	655.00	655	34.0	21.2	818	38	28
eilA101	37	26	3	65	60	650.80	4.7	231	180	657.00	657	39.7	25.3	1385	37	29
eilA101	36	28	3	70	63	652.01	4.8	402	525	657.00	657	174.9	88.3	4394	36	30
eilA101	35	30	3	90	84	652.79	5.7	638	997	661.00	661	595.1	283.1	17386	35	30
eilA101	34	32	3	90	85	653.50	5.2	1329	2059	662.50	665	2 hours	3205.4	197585	34	32
eilA101	28	16	4	54	47	672.08	3.2	149	113	674.00	674	15.0	10.5	432	28	20
eilA101	27	19	4	56	48	671.38	3.7	210	190	676.00	676	33.5	17.6	895	27	22
eilA101	26	22	4	89	84	675.00	4.8	537	1009	682.00	682	688.6	255.4	17968	26	24
eilA101	22	11	5	65	65	699.48	4.3	689	1416	704.00	704	730.4	179.3	9741	22	14
eilA101	21	16	5	69	71	698.91	4.5	1076	3189	709.07	712	2 hours	1527.4	120963	21	17
eilA101	18	10	6	81	85	725.28	5.6	1104	3622	734.87	741	2 hours	1521.5	98801	18	15
eilA101	17	15	6	69	78	727.73	4.9	1188	3581	735.23	805	2 hours	834.5	49901	17	15
A071-03f	37	33	2	19	29	1951.00	4.2	22	31	1979.00	1979	4.6	1.1	12	37	33
A071-03f	36	34	2	26	30	1954.00	0.8	37	42	1985.00	1985	1.1	0.4	21	35	35
A071-03f	35	35	2	21	27	1954.00	0.8	27	38	1985.00	1985	1.1	0.5	20	35	35
A071-03f	26	18	3	31	73	2031.00	1.3	99	743	2092.00	2092	32.3	17.5	1981	26	18
A071-03f	25	20	3	35	56	2029.33	1.1	133	815	2105.00	2105	47.0	24.4	3921	25	21
A071-03f	24	22	3	23	60	2034.81	1.1	115	1345	2105.00	2105	84.1	29.8	3619	24	22
A071-03f	20	10	4	25	101	2127.20	1.5	176	3155	2246.00	2246	582.6	107.5	18907	20	11
A071-03f	19	13	4	21	86	2147.17	1.7	321	7353	2257.00	2257	4882.6	302.5	37759	19	13
A071-03f	18	16	4	35	101	2165.54	1.7	207	6727	2281.00	2281	3397.6	280.8	42669	18	16
A071-03f	16	6	5	31	83	2290.00	1.4	269	5933	2406.00	2406	2262.5	150.8	21889	16	9
A071-03f	15	10	5	18	58	2290.71	1.3	283	8847	2439.00	2439	6381.9	294.5	48737	15	11

Table 8: CVRP with a fix number of vehicle (results with only CVRP inequalities).