On convergence in mixed integer programming

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Abstract

Let $P \subseteq \mathbb{R}^{m+n}$ be a rational polyhedron, and let $P_I := \text{conv}(P \cap (\mathbb{Z}^m \times \mathbb{R}^n))$ be the mixed integer hull of P. We define the *integral lattice-free closure* of P as the set obtained from P by adding all inequalities obtained from disjunctions associated with integral lattice-free sets in $\mathbb{R}^m$. We show that the integral lattice-free closure of P is again a polyhedron, and that repeatedly taking the integral lattice-free closure of P gives P_I after a finite number of iterations. Such results can be seen as a mixed integer analogue of theorems by Chvátal and Schrijver for the pure integer case.

One ingredient of our proof is an extension of a result by Owen and Mehrotra. In fact, we prove that for each rational polyhedron P, the split closures of P yield in the limit the set P_I.

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1 Introduction

Cutting plane techniques have been one of the prominent topics in the theory of integer and mixed integer programming. A fundamental result in the theory of cutting planes was shown by Chvátal [4] and Schrijver [12] [13, Theorem 23.2], and was inspired by Gomory's [6] early work. To state such result we recall the notion of Gomory cuts. An inequality $cx \leq \lfloor \gamma \rfloor$ is a *Gomory cut* for $P \subseteq \mathbb{R}^m$ if $c \in \mathbb{Z}^m$ and if $cx \leq \gamma$ is valid for P. The *Chvátal closure* P' of P is the set of all vectors that satisfy every Gomory cut for P. We denote by $P^{(i)}$, $i \in \mathbb{N}$ the *i-th Chvátal closure* of P, i.e. $P^{(i)} := (P^{(i-1)})'$, where $P^{(0)} := P$. A series of results of Chvátal and Schrijver gives the following theorem, where we denote by P_I the convex hull of the integral vectors in P, i.e. $P_I := \text{conv}(P \cap \mathbb{Z}^m)$.

Theorem 1. *For each rational polyhedron P, then*

(i) P' is again a rational polyhedron,

(ii) $P^{(k)} = P_I$ for some integer k.

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In a mixed integer programming problem, only a subset of the variables are restricted to integer values. Then the set of feasible solutions to such a problem attains the form

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \,:\, x \in \mathbb{Z}^m \right\}$$

where P is a polyhedron in $\mathbb{R}^{m+n}$. The vectors $\begin{pmatrix} x \\ y \end{pmatrix} \in P$ such that $x \in \mathbb{Z}^m$ are called *x-integral*. We denote by P_I the convex hull of the x-integral vectors in P, and we say that P is *x-integral* if $P = P_I$, i.e. if every minimal face of P contains x-integral vectors.

From Motzkin's decomposition theorem for polyhedra, it follows that if P is rational then P_I is a rational polyhedron (see [13, Section 16.7]).

It has been shown in [5] that a mixed integer problem with rational data can be artificially transformed into a pure integer problem at the expense of scaling up the data drastically. In fact this transformation is highly impractical, thus, it remained a challenge for many years to design finite cutting plane algorithms that directly work with the given formulation of a mixed integer problem and its corresponding space of variables.

An inequality $cx + dy \leq \gamma$ is a *split cut* for $P \subseteq \mathbb{R}^{m+n}$ if there exists a vector $a \in \mathbb{Z}^m$ and an integer β such that $cx + dy \leq \gamma$ is valid for both

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \,:\, ax \leq \beta \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \,:\, ax \geq \beta + 1 \right\}.$$

The *split closure* of P is defined as the set of all vectors that satisfy every split cut for P. Given $P \subseteq \mathbb{R}^{m+n}$, we denote by $\mathcal{S}(P)$ its split closure. Moreover, for every $i \in \mathbb{N}$, we denote by $\mathcal{S}^i(P)$ the *i-th split closure* of P, i.e. $\mathcal{S}^i(P) := \mathcal{S}(\mathcal{S}^{i-1}(P))$, where $\mathcal{S}^0(P) := P$. Cook et al. [5] proved that the split closure of a rational polyhedron $P \subseteq \mathbb{R}^{m+n}$ is again a rational polyhedron.

In the general mixed integer case, Cook et al. [5] showed that determining split closures does not suffice to generate P_I in a finite number of iterations. The reason is that, even if $\mathcal{S}^k(P) \neq P_I$ implies that $\mathcal{S}^{k+1}(P) \subsetneq \mathcal{S}^k(P)$, the difference between $\mathcal{S}^k(P)$ and $\mathcal{S}^{k+1}(P)$ may become arbitrarily small as k grows.

Split closures do not lead to finite convergence to P_I even if P is bounded. In this special case, however, Owen and Mehrotra [9] showed that the sequence of split closures does yield in the limit the set P_I.

In Section 2 we extend the result of Owen and Mehrotra from polytopes to rational polyhedra. That is, we prove that for each rational polyhedron P, the repeated computation of the split closures yields in the limit the set P_I. This result is a backbone of the main theorem of our paper, which can be seen as an analogue of Theorem 1 in mixed integer programming. In order to state it precisely, we next introduce the notion of lattice-free sets.

A polyhedron $L \subseteq \mathbb{R}^m$ is said to be *lattice-free* if $\mathrm{relint}(L) \cap \mathbb{Z}^m = \emptyset$. (We recall that the *relative interior* of a polyhedron L is the set of points x for which there exists a ball centered in x whose intersection with the affine hull of L is contained in L.) An inequality $cx + dy \leq \gamma$ is an *integral lattice-free cut* for $P \subseteq \mathbb{R}^{m+n}$ if there exists an integral lattice-free polyhedron $\{x \in \mathbb{R}^m \,:\, a_j x \leq$

$\beta_j, j = 1, \ldots, k\}$ such that $cx + dy \leq \gamma$ is valid for every set

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \,:\, a_j x \geq \beta_j \right\}, \quad j = 1, \ldots, k.$$

It is easy to see that an integral lattice-free cut is satisfied by all x-integral vectors in P. Clearly, every split cut for P is also an integral lattice-free cut for P. The *integral lattice-free closure* of P is defined as the set of all vectors that satisfy every integral lattice-free cut for P. Given $P \subseteq \mathbb{R}^{m+n}$, we denote by $\mathcal{L}(P)$ its integral lattice-free closure. Moreover, for every $i \in \mathbb{N}$, we denote by $\mathcal{L}^i(P)$ the *i-th integral lattice-free closure* of P, i.e. $\mathcal{L}^i(P) := \mathcal{L}(\mathcal{L}^{i-1}(P))$, where $\mathcal{L}^0(P) := P$.

We are now ready to state the main result of our paper. In Section 3 we prove that, if P is a rational polyhedron, then repeatedly taking the integral lattice-free closure of P gives P_I after a finite number of iterations. Moreover, we prove that the integral lattice-free closure of a rational polyhedron P is again a rational polyhedron.

From now on in this paper, if not explicitly stated, we work with rational spaces, rather than real ones. In particular, any matrix, any vector, and any polyhedron is supposed to be rational. Moreover we denote by $\mathbb{N}$ the set of nonnegative integers.

2 Infinite convergence

Let $\{\tilde{P}, P^i \,:\, i \in \mathbb{N}\}$ be a family of closed sets such that $\tilde{P} \subseteq P^{i+1} \subseteq P^i$ for every $i \in \mathbb{N}$. We say that the sequence $\{P^i \,:\, i \in \mathbb{N}\}$ *(Hausdorff) converges* to $\tilde{P}$ (i.e. $\lim_{i\to\infty} P^i = \tilde{P}$) if for every $\epsilon > 0$, there exists a $k \in \mathbb{N}$ such that $P^k \subseteq \tilde{P} + B(\epsilon)$, where $B(\epsilon)$ is the ball of radius ϵ, and the $+$ operator refers to the Minkowski sum. Notice that the given definition of convergence is based on the well-known Hausdorff distance. See [11] for more details. It is a well-known fact that if the sequnce $\{P^i \,:\, i \in \mathbb{N}\}$ converges, then $\lim_{i\to\infty} P^i = \bigcap_{i\in\mathbb{N}} P^i$.

The main result of this section is the following theorem.

Theorem 2. *For each rational polyhedron P*

$$\lim_{i\to\infty} \mathcal{S}^i(P) = P_I.$$

In the remainder of this section, we develop the proof of Theorem 2. Note that Theorem 2 in the special case when P is a polytope has been shown by Owen and Mehrotra [9]. We apply their overall proof strategy. However, several technical results are necessary to provide the proof for general polyhedra. We now state some well-known (and easy to prove) observations about Hausdorff convergence that are used in the proof.

Observation 1. *Let $\{\tilde{P}, \tilde{Q}, P^i, Q^i \,:\, i \in \mathbb{N}\}$ be a family of closed sets with $\tilde{P} \subseteq P^{i+1} \subseteq P^i$ for every $i \in \mathbb{N}$, $\tilde{Q} \subseteq Q^{i+1} \subseteq Q^i$ for every $i \in \mathbb{N}$, $\lim_{i\to\infty} P^i = \tilde{P}$, $\lim_{i\to\infty} Q^i = \tilde{Q}$. Then*

$$\lim_{i\to\infty} (P^i \cap Q^i) = \tilde{P} \cap \tilde{Q}, \quad \lim_{i\to\infty} (P^i \cup Q^i) = \tilde{P} \cup \tilde{Q}.$$

Observation 2. *Let $\{\tilde{P}, P^i : i \in \mathbb{N}\}$ be a family of closed sets with $\tilde{P} \subseteq P^{i+1} \subseteq P^i$ for every $i \in \mathbb{N}$, $\lim_{i\to\infty} P^i = \tilde{P}$, and such that the sets* $\mathrm{conv}P^i$ *and* $\mathrm{conv}\tilde{P}$ *are closed. Then*

$$\lim_{i\to\infty} \mathrm{conv}P^i = \mathrm{conv}\tilde{P}.$$

Observation 3. *Let $\{\tilde{P}, P^i : i \in \mathbb{N}\}$ be a family of closed sets such that $\tilde{P} \subseteq P^{i+1} \subseteq P^i$ for every $i \in \mathbb{N}$, where $\tilde{P} \neq \emptyset$ is the polyhedron defined by an irredundant system $a_j z \leq \beta_j$, $j = 1, \ldots, k$. Then the following are equivalent*

(i) $\lim_{i\to\infty} P^i = \tilde{P}$*;*

(ii) $\lim_{i\to\infty} \max\{a_j z : z \in P^i\} = \beta_j$ *for every* $j = 1, \ldots, k$*;*

(iii) $\lim_{i\to\infty} \max\{az : z \in P^i\} = \max\left\{az : z \in \tilde{P}\right\}$ *for every vector a for which the second maximum exists and is finite.*

One important ingredient of our proof is the following result about existence of a hyperplane which preserves mixed integrality under projection along a vector. To make this precise, we introduce the following notation. Given a nonzero vector $v \in \mathbb{Q}^{m+n}$, a subspace S of $\mathbb{R}^{m+n}$ of dimension $m + n - 1$ such that $v \notin S$, and a set W in $\mathbb{R}^{m+n}$, we denote by $\mathrm{proj}_{v,S}W$ the projection of W to the subspace S along the direction v, i.e. $\mathrm{proj}_{v,S}W = \{z \in S : z+\lambda v \in W, \exists \lambda \in \mathbb{R}\}$. Moreover we denote with $\mathrm{proj}_x W$ the orthogonal projection of W onto the space of the x-variables, i.e. $\mathrm{proj}_x W = \left\{x \in \mathbb{R}^m : \begin{pmatrix} x \\ y \end{pmatrix} \in W, \exists y \in \mathbb{R}^m\right\}$. Given a nonzero vector $v \in \mathbb{Q}^{m+n}$, and a subspace $\mathcal{H}$ of $\mathbb{R}^{m+n}$ of dimension $m + n - 1$, we say that $\mathcal{H}$ is *mixed integer invariant under projection along* v, if $v \notin \mathcal{H}$, and if $\mathrm{proj}_{v,\mathcal{H}} w \in \mathbb{Z}^m \times \mathbb{R}^n$ for every vector $w \in \mathbb{Z}^m \times \mathbb{R}^n$.

Lemma 4. *(Mixed integer invariance under projection) Let v be a nonzero vector in $\mathbb{Q}^{m+n}$. Then there exists a subspace $\mathcal{H}$ of $\mathbb{R}^{m+n}$ that is mixed integer invariant under projection along v.*

Proof. Let $v_x := \mathrm{proj}_x v$. If $v_x = 0$, then the result follows trivially by taking any subspace $\mathcal{H}$ of $\mathbb{R}^{m+n}$ of dimension $m+n-1$ such that $v \notin \mathcal{H}$. So we now assume $v_x \neq 0$. By scaling, we can assume that v_x is integral with $\gcd(v_1, \ldots, v_m) = 1$. Thus it is well-known (see for example Theorem 5 on page 21 in [7]) that there exists a lattice basis B of $\mathbb{Z}^m$ containing v_x. Let $\bar{\mathcal{H}} = \{x \in \mathbb{R}^m : hx = 0\}$ be the subspace of dimension $m - 1$ of $\mathbb{R}^m$ spanned by the vectors in $B \setminus \{v_x\}$. Clearly $v_x \notin \bar{\mathcal{H}}$, and since B is a basis of $\mathbb{Z}^m$, it follows that $\mathrm{proj}_{v_x,\bar{\mathcal{H}}}\bar{w} \in \mathbb{Z}^m$ for every vector $\bar{w} \in \mathbb{Z}^m$.

Now let $\mathcal{H} = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m+n} : hx = 0\right\}$. Clearly, the dimension of $\mathcal{H}$ is $m + n - 1$. Since $v_x \notin \bar{\mathcal{H}}$, then $hv_x \neq 0$, hence $v \notin \mathcal{H}$. Let $w \in \mathbb{Z}^m \times \mathbb{R}^n$, and let $w_x := \mathrm{proj}_x w$. We now show that $\mathrm{proj}_x(\mathrm{proj}_{v,\mathcal{H}} w) = \mathrm{proj}_{v_x,\bar{\mathcal{H}}}(w_x)$. Notice that $\mathrm{proj}_{v,\mathcal{H}}(w) = w + \alpha v$ for a scalar α, which implies that $hw_x + \alpha hv_x = 0$. This implies that $\mathrm{proj}_x(\mathrm{proj}_{v,\mathcal{H}}(w)) = w_x + \alpha v_x$. Since $hw_x + \alpha hv_x = 0$, then $w_x + \alpha v_x = \mathrm{proj}_{v_x,\bar{\mathcal{H}}}(w_x)$. Hence $\mathrm{proj}_x(\mathrm{proj}_{v,\mathcal{H}} w) = \mathrm{proj}_{v_x,\bar{\mathcal{H}}}(w_x)$. Since $w_x \in \mathbb{Z}^m$, it follows from the first part of the proof that $\mathrm{proj}_{v_x,\bar{\mathcal{H}}}(w_x) \in \mathbb{Z}^m$, thus $\mathrm{proj}_x(\mathrm{proj}_{v,\mathcal{H}} w) \in \mathbb{Z}^m$, which completes the proof. □

The next two lemmas establish important properties of the subspace $\mathcal{H}$ that is mixed integer invariant under projection along v. Their proofs are given in Section 4. We recall that the characteristic cone of a set P is $\text{char.cone}P := \{w : z + w \in P \text{ for all } z \in P\}$, and the lineality space of P is $\text{lin.space}P := \text{char.cone}P \cap -\text{char.cone}P$. A set P is called *pointed* if $\text{lin.space}P$ has dimension zero.

Lemma 5. *Let P be a polyhedron in $\mathbb{R}^{m+n}$, let v be a nonzero vector, and let $\mathcal{H}$ be a subspace that is mixed integer invariant under projection along v. Then*

$$\text{proj}_{v,\mathcal{H}}\mathcal{S}^i(P) \subseteq \mathcal{S}^i(\text{proj}_{v,\mathcal{H}}P) \quad \textit{for every } i \in \mathbb{N}.$$

Lemma 6. *Let P be an unbounded polyhedron in $\mathbb{R}^{m+n}$, let v be a nonzero vector in* $\text{char.cone}P$*, and let $\mathcal{H}$ be a subspace that is mixed integer invariant under projection along v. Then*

$$\text{proj}_{v,\mathcal{H}}(P_I) = (\text{proj}_{v,\mathcal{H}}P)_I.$$

We are now prepared to prove the main result of this section.

Proof of Theorem 2. Let $P \subseteq \mathbb{R}^{m+n}$ be a polyhedron, and notice that $P_I \subseteq \mathcal{S}^{i+1}(P) \subseteq \mathcal{S}^i(P)$ for every $i \in \mathbb{N}$. Moreover by Cook et. al [5], $\{P_I, S^i(P) : i \in \mathbb{N}\}$ is a family of polyhedra, thus of closed sets. The proof is by induction on the dimension of P, the cases $\dim P = -1$ (i.e. $P = \emptyset$) and $\dim P = 0$ (i.e. P is a singleton) being trivial. If P is bounded, then the result follows from Owen and Mehrotra [9]. Thus, from now on, we assume that P is unbounded.

Claim 1. *If* $\text{lin.space}P \neq \{0\}$*, then* $\lim_{i\to\infty} \mathcal{S}^i(P) = P_I$.

Proof of claim. Let v be a nonzero vector in $\text{lin.space}P$. By Lemma 4 there exists a subspace $\mathcal{H}$ of $\mathbb{R}^{m+n}$ that is mixed integer invariant under projection along v. Let $\bar{P} = \text{proj}_{v,\mathcal{H}}P = P \cap \mathcal{H}$. Since v is a nonzero vector in $\text{lin.space}P$, then $\dim \bar{P} < \dim P$. Thus by the hypothesis of the induction the sequence $\{\mathcal{S}^i(\bar{P}) : i \in \mathbb{N}\}$ converges to $\bar{P}_I$. Clearly the sequence $\{\mathcal{S}^i(\bar{P}) + \text{lin.hull}\{v\} : i \in \mathbb{N}\}$ converges to $\bar{P}_I + \text{lin.hull}\{v\}$. By Lemma 6, $\text{proj}_{v,\mathcal{H}}(P_I) = \bar{P}_I$, and, since $v \in \text{lin.space}(P_I)$, it follows that $P_I = \bar{P}_I + \text{lin.hull}\{v\}$. Hence the sequence $\{\mathcal{S}^i(\bar{P}) + \text{lin.hull}\{v\} : i \in \mathbb{N}\}$ converges to P_I. By Lemma 5, $\text{proj}_{v,\mathcal{H}}\mathcal{S}^i(P) \subseteq \mathcal{S}^i(\bar{P})$ for every $i \in \mathbb{N}$, which implies that $\mathcal{S}^i(P) \subseteq \mathcal{S}^i(\bar{P}) + \text{lin.hull}\{v\}$ for every $i \in \mathbb{N}$. Since moreover $P_I \subseteq \mathcal{S}^i(P)$ for every $i \in \mathbb{N}$, it follows that $\lim_{i\to\infty} \mathcal{S}^i(P) = P_I$. $\diamond$

Thus, from now on, we can assume that P is unbounded and pointed.

Claim 2. *If $P_I = \emptyset$, then* $\lim_{i\to\infty} \mathcal{S}^i(P) = \emptyset$.

Proof of claim. Let $v \in \text{char.cone}P$, and let $\bar{P} = P + \text{lin.hull}\{v\}$. Clearly $P \subseteq \bar{P}$, $\dim \bar{P} = \dim P$ and $\text{lin.space}\bar{P} \neq \emptyset$. Moreover it is straightforward to verify that $\bar{P}_I = P_I = \emptyset$. By Claim 1, $\lim_{i\to\infty} \mathcal{S}^i(\bar{P}) = \emptyset$. Since $P \subseteq \bar{P}$, then $\mathcal{S}^i(P) \subseteq \mathcal{S}^i(\bar{P})$ for every $i \in \mathbb{N}$. Thus $\lim_{i\to\infty} \mathcal{S}^i(P) \subseteq \lim_{i\to\infty} \mathcal{S}^i(\bar{P}) = \emptyset$. $\diamond$

From now on we can assume that P is unbounded, pointed, and such that $P_I \neq \emptyset$. Notice that this implies that $\mathcal{S}^i(P) \neq \emptyset$ for all $i \in \mathbb{N}$. In the remainder of the proof, we use the following basic fact:

$$\text{char.cone}\mathcal{S}^i(P) = \text{char.cone}P = \text{char.cone}(P_I) \quad \text{for every } i \in \mathbb{N}.$$

We define

$$C := \text{char.cone}(P).$$

Claim 3. *The sequence $\{\mathcal{S}^i(P) : i \in \mathbb{N}\}$ converges.*

Proof of claim. Since P is pointed and $P_I \neq \emptyset$, it follows that also P_I is pointed. Hence it is easy to see that there exists a half-space

$$Q^{\geq} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m+n} : q_x x + q_y y \geq \rho \right\}$$

that contains every vertex of P_I, and such that $P \cap Q^{\geq}$ is bounded. Since $\{\mathcal{S}^i(P) : i \in \mathbb{N}\}$ is a sequence of polyhedra, it follows that $\{\mathcal{S}^i(P) \cap Q^{\geq} : i \in \mathbb{N}\}$ is a sequence of compact sets, and they are all contained in the polytope $P \cap Q^{\geq}$. Hence it follows from the Blaschke selection theorem [3] that $\{\mathcal{S}^i(P) \cap Q^{\geq} : i \in \mathbb{N}\}$ converges to some compact set $\hat{P} \subseteq P \cap Q^{\geq}$.

Now let

$$Q^{\leq} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m+n} : q_x x + q_y y \leq \rho \right\}.$$

We now show that $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap Q^{\leq}) = P_I \cap Q^{\leq}$. Notice that, since all the vertices of P_I are contained in $Q^{\geq}$, then an irredundant inequality description of $P_I \cap Q^{\leq}$ is given by the inequality $q_x x + q_y y \leq \rho$, by a system of equations defining the affine hull of P_I, and by the inequalities defining the unbounded facets of P_I. By Observation 3 (ii), to prove that $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap Q^{\leq}) = P_I \cap Q^{\leq}$, we only need to prove that $\lim_{i\to\infty} \max\left\{ cx + dy : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^i(P) \cap Q^{\leq} \right\} = \gamma$ for every constraint $cx + dy \leq \gamma$ of such irredundant inequality description of $P_I \cap Q^{\leq}$.

Since such property is valid by definition for the constraint $q_x x + q_y y \leq \rho$, we can assume that the vector (c, d) is such that $cx + dy \leq \gamma$ is valid for P_I, and $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P_I : cx + dy = \gamma \right\}$ is unbounded. Since $\mathcal{S}^i(P) \cap Q^{\leq} \subseteq \mathcal{S}^i(P)$ for every $i \in \mathbb{N}$, we only need to verify that, for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $cx + dy \leq \gamma + \epsilon$ is valid for $\mathcal{S}^k(P)$. Now let $\epsilon > 0$, and let v be a nonzero vector in $\text{char.cone}\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P_I : cx + dy = \gamma \right\}$. Clearly $v \in C$. By Lemma 4 there exists a subspace $\mathcal{H}$ of $\mathbb{R}^{m+n}$ that is mixed integer invariant under projection along v. Let $\bar{P} = \text{proj}_{v,\mathcal{H}} P$. Since v is a nonzero vector in C, then $\dim(\bar{P}) < \dim(P)$. Thus by induction the sequence $\{\mathcal{S}^i(\bar{P}) : i \in \mathbb{N}\}$ converges to $\bar{P}_I$. Since $cx + dy \leq \gamma$ is valid for P_I, and since $(c, d)v = 0$, it follows that $cx + dy \leq \gamma$ is also valid for $\text{proj}_{v,\mathcal{H}}(P_I)$. By Lemma 6, $\text{proj}_{v,\mathcal{H}}(P_I) = \bar{P}_I$, hence $cx + dy \leq \gamma$ is valid for $\bar{P}_I$. Since $\{\mathcal{S}^i(\bar{P}) : i \in \mathbb{N}\}$ converges to $\bar{P}_I$, there exists $k \in \mathbb{N}$ such that $cx + dy \leq \gamma + \epsilon$ is valid for $\mathcal{S}^k(\bar{P})$. By Lemma 5, $\text{proj}_{v,\mathcal{H}}\mathcal{S}^k(P) \subseteq \mathcal{S}^k(\bar{P})$, thus $cx + dy \leq \gamma + \epsilon$ is valid for $\text{proj}_{v,\mathcal{H}}\mathcal{S}^k(P)$. Finally, since $(c, d)v = 0$, it follows that $cx + dy \leq \gamma + \epsilon$ is valid for $\mathcal{S}^k(P)$. Hence we showed that $\lim_{i\to\infty}(\mathcal{S}^i(P) \cap Q^{\leq}) = P_I \cap Q^{\leq}$.

Since both sequences $\{\mathcal{S}^i(P) \cap Q^{\geq} : i \in \mathbb{N}\}$ and $\{\mathcal{S}^i(P) \cap Q^{\leq} : i \in \mathbb{N}\}$ converge, then by Observation 1 we conclude that also the sequence $\{\mathcal{S}^i(P) : i \in \mathbb{N}\}$ converges. $\diamond$

Thus $\{\mathcal{S}^i(P) : i \in \mathbb{N}\}$ converges to some closed set $\tilde{P}$ such that $P_I \subseteq \tilde{P}$. By Observation 1, this implies that $\lim_{i\to\infty} \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^i(P) : ax \le \beta \right\} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \tilde{P} : ax \le \beta \right\}$ for any vector a and scalar β. Now let $a \in \mathbb{Z}^m$, $\beta \in \mathbb{Z}$. Then, by definition of split closure,

$$\mathcal{S}^{i+1}(P) \subseteq \operatorname{conv}\left(\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^i(P) : ax \le \beta \right\} \cup \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^i(P) : ax \ge \beta + 1 \right\} \right).$$

Taking the limit and applying Observation 2 and Observation 1, we have that

$$\begin{aligned}
\tilde{P} &= \lim_{i\to\infty} \mathcal{S}^{i+1}(P) \\
&\subseteq \lim_{i\to\infty} \operatorname{conv}\left(\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^i(P) : ax \le \beta \right\} \cup \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^i(P) : ax \ge \beta + 1 \right\} \right) \\
&= \operatorname{conv} \lim_{i\to\infty} \left(\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^i(P) : ax \le \beta \right\} \cup \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^i(P) : ax \ge \beta + 1 \right\} \right) \\
&= \operatorname{conv} \left(\lim_{i\to\infty} \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^i(P) : ax \le \beta \right\} \cup \lim_{i\to\infty} \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^i(P) : ax \ge \beta + 1 \right\} \right) \\
&= \operatorname{conv}\left(\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \tilde{P} : ax \le \beta \right\} \cup \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \tilde{P} : ax \ge \beta + 1 \right\} \right).
\end{aligned}$$

Hence,

$$\tilde{P} \subseteq \bigcap_{(a,\beta)\in\mathbb{Z}^{m+1}} \operatorname{conv}\left(\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \tilde{P} : ax \le \beta \right\} \cup \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \tilde{P} : ax \ge \beta + 1 \right\} \right),$$

which means that the split closure of $\tilde{P}$ is equal to $\tilde{P}$.

We now show that this implies

$$\tilde{P} = P_I.$$

We prove this by contradiction, thus assume $P_I \subsetneq \tilde{P}$. Notice that $\tilde{P}$ is convex (as it is an intersection of convex sets), closed, and pointed (since P is pointed). It follows that there exists a full-dimensional cone C in $\mathbb{R}^{m+n}$ such that $\max\{ez : z \in \tilde{P}\} \gneq \max\{ez : z \in \tilde{P}_I\}$ for every nonzero vector $e \in C$. Let $\tilde{C}$ be the open set obtained from C by removing its boundary. Now let f be the function defined by $f(e) = \max\{ez : z \in \tilde{P}\}$ for every $e \in \tilde{C}$, and notice that f is strictly continuous. By Rademacher's theorem [10, Theorem 9.60] there exists $\tilde{e} \in \tilde{C}$ such that f is differentiable in $\tilde{c}$. It follows that $\max\{\tilde{e}z : z \in \tilde{P}\}$ is achieved by only one vector $\tilde{z} \in \tilde{P}$. By definition of $\tilde{C}$ it follows that $\tilde{z} \notin \mathbb{Z}^m \times \mathbb{R}^n$. It is now easy to construct a split cut for $\tilde{P}$ that cuts the vector $\tilde{z}$, which implies $\mathcal{S}(\tilde{P}) \subsetneq \tilde{P}$, a contradiction. □

3 Finite convergence

In this section we prove an analogue of Theorem 1 in mixed integer programming. Indeed, we show that the integral lattice-free closure of a polyhedron is again a polyhedron, and that repeatedly taking the integral lattice-free closure of P gives P_I after a finite number of iterations.

In [5] it was shown that the split closure of a polyhedron is again a polyhedron. This result carries over to the closure of a polyhedron obtained from disjunctions associated with any family of integral lattice-free polyhedra. This follows quite easily combining the results in [1, 2]. For the remainder of the paper the following specialization is of importance.

Theorem 3. *For any rational polyhedron P, the set $\mathcal{L}(P)$ is again a rational polyhedron.*

Proof. Let P be a polyhedron in $\mathbb{R}^{m+n}$. It follows by definition of integral lattice-free cut, that each integral lattice-free cut for P, which is not a valid inequality for P, corresponds to an integral lattice-free polyhedron of $\mathbb{R}^m$ of dimension m. Given two lattice-free polyhedra L,L' in $\mathbb{R}^m$ of dimension m and with $L \subseteq L'$, then clearly $\mathrm{relint} L \subseteq \mathrm{relint} L'$. It follows that each irredundant integral lattice-free cut for P corresponds to an integral lattice-free polyhedron in $\mathbb{R}^m$ of dimension m which is maximal with respect to inclusion. Thus let Z be the set of such maximal lattice-free polyhedra.

A result by Averkov et al. [2] implies that the set Z is finite up to affine unimodular transformations (i.e. transformations that map $\mathbb{Z}^m$ onto $\mathbb{Z}^m$). Moreover it is well-known that such transformations preserve the facet widths of any polyhedron. This implies that Z is a family of lattice-free polyhedra of bounded max-facet-width (see [1] for details). By Theorem 4.3 in [1] this implies that the set $\mathcal{L}(P)$ is a rational polyhedron. □

The main result of this paper is the following.

Theorem 4. *For each rational polyhedron P there exists a number k such that*

$$\mathcal{L}^k(P) = P_I.$$

Our proof of Theorem 4 is quite technical. It requires two Lemmas that we state first. In order to streamline the presentation, we postpone the proof of Lemma 7 to Section 4. At this point, let us just mention that Lemma 7 applied to polytopes has already been proven by Jörg [8]. We adapt his proof technique to show the general result.

Lemma 7. *Let P be a polyhedron in $\mathbb{R}^{m+n}$, and let $(c, d) \in \mathbb{Q}^{m+n}$ such that $\gamma := \max\left\{cx + dy \,:\, \begin{pmatrix} x \\ y \end{pmatrix} \in P_I\right\}$ is finite. If the set*

$$\mathrm{proj}_x \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P_I \,:\, cx + dy = \gamma \right\}$$

is not lattice-free, then $\exists k \in \mathbb{N}$ with $\max\left\{cx + dy \,:\, \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^k(P)\right\} = \gamma$.

Lemma 8. *Let P be an integral lattice-free polyhedron in $\mathbb{R}^m$. Then there exists an integral lattice-free polyhedron $L \supseteq P$ of dimension m such that $\mathrm{relint} P \subseteq \mathrm{relint} L$.*

Proof. If $\dim P = m$ then $L = P$, so we now assume $\dim P < m$. Since the lemma is invariant under affine unimodular transformations (i.e. transformations that map $\mathbb{Z}^m$ onto $\mathbb{Z}^m$), we may assume that $\text{aff.hull} P = \text{lin.hull}\{e_1, \ldots, e_{\dim P}\}$. It is then easy to verify that $L := P + \text{lin.hull}\{e_{\dim P+1}, \ldots, e_m\}$ is a lattice-free polyhedron of dimension m with $\text{relint} P \subseteq \text{relint} L$. □

We now prove our main result.

Proof of Theorem 4. Let $P \subseteq \mathbb{R}^{m+n}$ be a polyhedron. If $P_I = \emptyset$, then by Theorem 2, $\lim_{i\to\infty} \mathcal{S}^i(P) = \emptyset$, which implies that there exists a number $k \in \mathbb{N}$ such that $\mathcal{S}^k(P) = \emptyset$. Since $\mathcal{L}^k(P) \subseteq \mathcal{S}^k(P)$, it follows that $\mathcal{L}^k(P) = \emptyset$. Thus, we now assume that $P_I \neq \emptyset$.

Since $P_I \neq \emptyset$, then to prove the theorem, we need to show that for every vector $(c, d) \in \mathbb{Q}^{m+n}$ such that $\gamma := \max \left\{ cx + dy \, : \, \begin{pmatrix} x \\ y \end{pmatrix} \in P_I \right\}$ is finite, there exists $k \in \mathbb{N}$ with $\max \left\{ cx + dy \, : \, \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{L}^k(P) \right\} = \gamma$. We prove this by induction on $\dim F$, where $F := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P_I \, : \, cx + dy = \gamma \right\}$.

If F is a minimal face of P_I, then F is an affine space and it contains x-integral vectors. It follows that $\text{proj}_x F$ is an affine space too, and it contains integral vectors. Since $\text{proj}_x F$ is an affine space, and the relative interior of every affine space is the same affine space, then $\text{proj}_x F$ contains integral vectors in its relative interior, thus it is not lattice-free. Then the statement follows from Lemma 7, as each split cut is also an integral lattice-free cut.

So now let F be a face of P_I which is not minimal. If $\text{proj}_x F$ is not lattice-free, then the statement follows from Lemma 7. Hence, we now assume that $\text{proj}_x F$ is lattice-free.

Since $\text{proj}_x F$ is an integral lattice-free polyhedron, it follows by Lemma 8 there exists an integral lattice-free polyhedron $L \subseteq \mathbb{R}^m$ of dimension m such that $\text{proj}_x F \subseteq L$ and $\text{relint}(\text{proj}_x F) \subseteq \text{relint} L$. Then the result follows, if we show that there exists a $k \in \mathbb{N}$ such that

$$\text{proj}_x \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{L}^k(P) \, : \, cx + dy > \gamma \right\} \subseteq \text{relint} L.$$

Now let $a_j x \leq \beta_j$, $j \in J$, be a minimal system of inequalities defining L.

Claim 4. *For every $j \in J$, there exists an inequality $c_j x + dy \leq \gamma_j$ such that:*

(i) $c_j x + dy \leq \gamma_j$ is valid for P_I;

(ii) $F_j := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P_I \, : \, c_j x + dy = \gamma_j \right\} \subsetneq F$;

(iii) $a_j x < \beta_j$ is valid for $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m+n} \, : \, cx + dy > \gamma, c_j x + dy \leq \gamma_j \right\}$.

Proof of claim. Let $j \in J$. For every $\epsilon \geq 0$, consider the inequality $(\epsilon a_j + c)x + dy \leq \epsilon\beta_j + \gamma$.

(i). Notice that, since $\max \left\{ cx + dy \, : \, \begin{pmatrix} x \\ y \end{pmatrix} \in P_I \right\}$ is attained in the face F, since $a_j x \leq \beta_j$ is valid for F, and since $\lim_{\epsilon\to 0} \epsilon a_j = 0$, it follows that there exists

$\bar{\epsilon} > 0$ small enough such that $\max\left\{(\bar{\epsilon}a_j + c)x + dy \,:\, \begin{pmatrix} x \\ y \end{pmatrix} \in P_I\right\}$ is attained in a face of F. Let $c_j := \bar{\epsilon}a_j + c$, and $\gamma_j := \bar{\epsilon}\beta_j + \gamma$. Since both inequalities $cx + dy \leq \gamma$, and $a_j x \leq \beta_j$ are valid for F, it follows that also their conic combination $c_j x + dy \leq \gamma_j$ is valid for F. Since $\max\left\{c_j x + dy \,:\, \begin{pmatrix} x \\ y \end{pmatrix} \in P_I\right\}$ is attained in a face of F, and $c_j x + dy \leq \gamma_j$ is valid for F, then $c_j x + dy \leq \gamma_j$ is valid for P_I.

(ii). Since $c_j x + dy \leq \gamma_j$ is valid for P_I, and $\max\left\{c_j x + dy \,:\, \begin{pmatrix} x \\ y \end{pmatrix} \in P_I\right\}$ is attained in a face of F, then $F_j \subseteq F$ (notice that F_j can also be the emptyset). To prove that the inclusion is proper, let $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \in F$ with $\bar{x} \in \text{relint}(\text{proj}_x F)$. Since $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \in F$, then $c\bar{x} + d\bar{y} \leq \gamma$. Moreover, since $\bar{x} \in \text{relint}(\text{proj}_x F)$, then $\bar{x} \in \text{relint} L$, hence $a_j \bar{x} < \beta_j$. Since $\bar{\epsilon} > 0$, it follows that $(\bar{\epsilon}a_j + c)\bar{x} + d\bar{y} < \bar{\epsilon}\beta_j + \gamma$. Hence $\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} \in F$ does not satisfy $c_j x + dy = \gamma_j$, implying $F_j \subsetneq F$.

(iii). Follows by definition of the inequality $c_j x + dy \leq \gamma_j$, and the fact that $\bar{\epsilon} > 0$. $\diamond$

For each $j \in J$, let $c_j x + dy \leq \gamma_j$ be an inequality as in Claim 4. We show next that for every $j \in J$, there exists a $k \in \mathbb{N}$ such that the inequality $c_j x + dy \leq \gamma_j$ is valid for $\mathcal{L}^k(P)$. By Claim 4(i) $c_j x + dy \leq \gamma_j$ is valid for P_I. Hence, if $F_j = \emptyset$, then by Theorem 2 there exists $k \in \mathbb{N}$ such that $c_j x + dy \leq \gamma_j$ is valid for $\mathcal{S}^k(P)$, and so for $\mathcal{L}^k(P)$. Otherwise $F_j \neq \emptyset$. In this case it follows by Claim 4(ii) that the set F_j is a proper face of F, which implies that $\dim F_j < \dim F$. Thus by hypothesis of the induction, there exists $k \in \mathbb{N}$ such that $c_j x + dy \leq \gamma_j$ is valid for $\mathcal{L}^k(P)$.

Since J is finite, it follows that there exists $k \in \mathbb{N}$ such that all the inequalities $c_j x + dy \leq \gamma_j$, $j \in J$, are valid for $\mathcal{L}^k(P)$.

Finally, by applying Claim 4(iii) to every $j \in J$, it follows that $\text{proj}_x Q \subseteq \text{relint}(L)$, where

$$Q := \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{m+n} \,:\, cx + dy > \gamma, c_j x + dy \leq \gamma_j \;\forall j \in J\right\}.$$

Since $\left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{L}^k(P) \,:\, cx + dy > \gamma\right\} \subseteq Q$, it follows that

$$\text{proj}_x \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{L}^k(P) \,:\, cx + dy > \gamma\right\} \subseteq \text{relint}(L).$$

□

4 Proofs of technical lemmas

Proof of Lemma 5. The proof is by induction on $i \geq 0$, the case $i = 0$ being trivial. We now show the case $i = 1$, i.e. that $\text{proj}_{v,\mathcal{H}}\mathcal{S}(P) \subseteq \mathcal{S}(\text{proj}_{v,\mathcal{H}}P)$.

Let $\bar{P} := \text{proj}_{v,\mathcal{H}}P$, and let $\bar{z} = \begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} \notin \mathcal{S}(\bar{P})$. We want to show that $\bar{z} \notin \text{proj}_{v,\mathcal{H}}\mathcal{S}(P)$. If $\bar{z} \notin \bar{P}$, then clearly $\bar{z} \notin \text{proj}_{v,\mathcal{H}}\mathcal{S}(P)$, as $\mathcal{S}(P) \subseteq P$. So we now assume $\bar{z} \in \bar{P}$. Thus there exists a split cut $cx + dy \leq \gamma$ for $\bar{P}$ such that $c\bar{x} + d\bar{y} > \gamma$. This implies that there exist $a \in \mathbb{Z}^m$, $\beta \in \mathbb{Z}$ such that $cx + dy \leq \gamma$ is valid for both $\bar{P} \cap H_1$ and $\bar{P} \cap H_2$, where $H_1 = \left\{\begin{pmatrix}x\\ y\end{pmatrix} \in \mathbb{R}^{m+n} \,:\, ax \leq \beta\right\}$, $H_2 = \left\{\begin{pmatrix}x\\ y\end{pmatrix} \in \mathbb{R}^{m+n} \,:\, ax \geq \beta + 1\right\}$.

Now let $c'x + d'y \leq \gamma'$ be the inequality defining the half-space

$$\left\{\begin{pmatrix}x\\ y\end{pmatrix} \in \mathcal{H} \,:\, cx + dy \leq \gamma\right\} + \text{lin.hull}\{v\}.$$

Notice that by construction $(c', d')v = 0$, and $\left\{\begin{pmatrix}x\\ y\end{pmatrix} \in \mathcal{H} \,:\, cx + dy \leq \gamma\right\} = \left\{\begin{pmatrix}x\\ y\end{pmatrix} \in \mathcal{H} \,:\, c'x + d'y \leq \gamma'\right\}$. Moreover, let $H_i' := (H_i \cap \mathcal{H}) + \text{lin.hull}\{v\}$ for $i \in \{1, 2\}$, and notice that $H_i' \cap \mathcal{H} = H_i \cap \mathcal{H}$ for $i \in \{1, 2\}$. Since every x-integral vector is contained in $H_1 \cup H_2$, it follows that every x-integral vector in $\mathcal{H}$ is in $H_1' \cup H_2'$. Since $\mathcal{H}$ is mixed integer invariant under projection along v, then by definition of H_1', H_2', it follows that every x-integral vector is contained in $H_1' \cup H_2'$. Thus there exist $a' \in \mathbb{Z}^m$, $\beta' \in \mathbb{Z}$ such that $Q_1 \subseteq H_1'$, and $Q_2 \subseteq H_2'$, where $Q_1 := \left\{\begin{pmatrix}x\\ y\end{pmatrix} \in \mathbb{R}^{m+n} \,:\, a'x \leq \beta'\right\}$, and $Q_2 := \left\{\begin{pmatrix}x\\ y\end{pmatrix} \in \mathbb{R}^{m+n} \,:\, a'x \geq \beta' + 1\right\}$.

Since $\bar{z}$ and $\bar{P}$ lie in $\mathcal{H}$, it follows that $c'\bar{x} + d'\bar{y} > \gamma'$ and that $c'x + d'y \leq \gamma'$ is valid for both $\bar{P} \cap H_1'$ and $\bar{P} \cap H_2'$. Since $(c', d')v = 0$, it follows that $c'x + d'y \leq \gamma'$ is valid for both $P \cap H_1'$ and $P \cap H_2'$, and so for both $P \cap Q_1$ and $P \cap Q_2$. Hence $c'x + d'y \leq \gamma'$ is a split cut for P with $c'\bar{x} + d'\bar{y} > \gamma'$, implying that $\bar{z} \notin \mathcal{S}(P)$. Since $(c', d')v = 0$, it follows that $\bar{z} \notin \text{proj}_{v,\mathcal{H}}\mathcal{S}(P)$. Thus we showed $\text{proj}_{v,\mathcal{H}}\mathcal{S}(P) \subseteq \mathcal{S}(\text{proj}_{v,\mathcal{H}}P)$.

Now for $i \geq 2$,

$$\begin{aligned}\text{proj}_{v,\mathcal{H}}\mathcal{S}^i(P) = \text{proj}_{v,\mathcal{H}}\mathcal{S}(\mathcal{S}^{i-1}(P)) \subseteq \mathcal{S}(\text{proj}_{v,\mathcal{H}}\mathcal{S}^{i-1}(P)) \subseteq \\ \subseteq \mathcal{S}(\mathcal{S}^{i-1}(\text{proj}_{v,\mathcal{H}}P)) = \mathcal{S}^i(\text{proj}_{v,\mathcal{H}}P).\end{aligned}$$

□

Proof of Lemma 6. Let $\bar{P} = \text{proj}_{v,\mathcal{H}}P$. Clearly $\bar{P}_I$ is x-integral, and, since P_I is x-integral and $\mathcal{H}$ is mixed integer invariant under projection along v, it follows that also $\text{proj}_{v,\mathcal{H}}(P_I)$ is x-integral. Thus we only need to show that $\text{proj}_{v,\mathcal{H}}(P_I) \cap \mathbb{Z}^m \times \mathbb{R}^n = \bar{P}_I \cap \mathbb{Z}^m \times \mathbb{R}^n$.

Let $z \in \text{proj}_{v,\mathcal{H}}(P_I) \cap \mathbb{Z}^m \times \mathbb{R}^n$. Clearly $z \in \bar{P}$, and since z is x-integral, it follows that $z \in \bar{P}_I$.

To prove the converse, let $z \in \bar{P}_I \cap \mathbb{Z}^m \times \mathbb{R}^n$. Since in particular $z \in \bar{P}$, it follows there exists a scalar λ such that $z + \lambda v \in P$. As z is x-integral, and v is rational, it follows that there exists a scalar $\mu \geq 0$ such that $w = z + \lambda v + \mu v$ is x-integral. Since $z + \lambda v \in P$, and v is a nonzero vector in $\text{char.cone}P$, it follows that $w \in P$. This implies $w \in P_I$ and so $z = \text{proj}_{v,\mathcal{H}}w \in \text{proj}_{v,\mathcal{H}}(P_I)$. □

Proof of Lemma 7. We define the sets $F := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P_I \,:\, cx + dy = \gamma \right\}$ and $Q := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \,:\, cx + dy > \gamma \right\}$. If $Q = \emptyset$ there is nothing to show, thus we assume $Q \neq \emptyset$. Let $\bar{Q} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in P \,:\, cx + dy \geq \gamma \right\}$ be the closure of Q. Since $cx + dy \leq \gamma$ is valid for P_I, then Q does not contain any x-integral vector. It follows that $(\text{proj}_x Q) \cap \mathbb{Z}^m = \emptyset$, and that $\text{proj}_x \bar{Q}$ is lattice-free, since it is the closure of $\text{proj}_x Q$.

Now we show that there exists an inequality $ax \geq \beta$, with $a \in \mathbb{Z}^m$, $\beta \in \mathbb{Z}$, such that $ax = \beta$ is valid for $\text{proj}_x F$, and $ax > \beta$ is valid for $\text{proj}_x Q$. Since $F \subseteq \bar{Q}$, then $\text{proj}_x F \subseteq \text{proj}_x \bar{Q}$, and since $\text{proj}_x \bar{Q}$ is lattice-free while $\text{proj}_x F$ is not, then $\dim(\text{proj}_x F) < \dim(\text{proj}_x \bar{Q})$ and $\text{proj}_x F$ is contained in a proper face of $\text{proj}_x \bar{Q}$. Let G be a minimal face of $\text{proj}_x \bar{Q}$, with respect to inclusion, containing $\text{proj}_x F$. It follows that G is not lattice-free. Now let $ax \geq \beta$ be valid for $\text{proj}_x \bar{Q}$ and such that $G = \{x \in \text{proj}_x \bar{Q} \,:\, ax = \beta\}$. Then clearly $ax = \beta$ is valid for $\text{proj}_x F$. Moreover, since $\text{proj}_x Q$ contains no integral point, and since G is not lattice-free, then it is easy to verify that $\text{proj}_x Q$ satisfies $ax > \beta$. Clearly, since a is rational and $\text{proj}_x F$ contains integral vectors, we can assume that $a \in \mathbb{Z}^m$ and $\beta \in \mathbb{Z}$.

We now complete the proof by showing that there exists a split closure $\mathcal{S}^k(P)$, $k \in \mathbb{N}$, of P such that $cx + dy \leq \gamma$ is valid for both

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^k(P) \,:\, ax \leq \beta \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^k(P) \,:\, ax \geq \beta + 1 \right\}.$$

We now introduce the sets $Q^i := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^i(P) \,:\, cx + dy > \gamma \right\}$ and $\bar{Q}^i := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{S}^i(P) \,:\, cx + dy \geq \gamma \right\}$ for every $i \in \mathbb{N}$. By Theorem 2, $\lim_{i \to \infty} \mathcal{S}^i(P) = P_I$. By intersecting P_I and $\mathcal{S}^i(P)$ for every $i \in \mathbb{N}$ with the half-space corresponding to the inequality $cx + dy \geq \gamma$, it follows that $\lim_{i \to \infty} \bar{Q}^i = F$. It is a well-known fact that this implies also convergence on the projections, i.e. $\lim_{i \to \infty} \text{proj}_x \bar{Q}^i = \text{proj}_x F$. Hence by Observation 3 (iii):

$$\lim_{i \to \infty} \max\{ax \,:\, x \in \text{proj}_x \bar{Q}^i\} = \beta.$$

Thus there exists a $k \in \mathbb{N}$ such that

$$\beta \leq ax < \beta + 1 \text{ for all } x \in \bar{Q}^k.$$

Since $Q^i \subseteq \bar{Q}^i$ for every $i \in \mathbb{N}$, it follows that

$$\beta \leq ax < \beta + 1 \text{ for all } x \in Q^k.$$

Finally, since $Q^i \subseteq Q$ for every $i \in \mathbb{N}$, and since Q satisfies $ax > \beta$, then

$$\beta < ax < \beta + 1 \text{ for all } x \in Q^k,$$

implying that $cx + dy \leq \gamma$ is a split cut for $\mathcal{S}^k(P)$. $\square$

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References

[1] K. Andersen, Q. Louveaux, R. Weismantel, An analysis of mixed integer linear sets based on lattice point free convex sets, *Mathematics of Operations Research* **35** (2010) 233-256.

[2] G. Averkov, C. Wagner, R. Weismantel, Maximal lattice-free polyhedra: finiteness and an explicit description in dimension three, http://arxiv.org/abs/1010.1077 (2010).

[3] W. Blaschke, *Kreis und Kugel*, Chelsea, New York (1949).

[4] V. Chvátal, Edmonds polytopes and a hierarchy of combinatorial problems, *Discrete Mathematics* **4** (1973) 305-337.

[5] W.J. Cook, R. Kannan, A. Schrijver, Chvátal closures for mixed integer programming problems, *Mathematical Programming* **47** (1990) 155-174.

[6] R.E. Gomory, An algorithm for integer solutions to linear programs, in *Recent Advances in Mathematical Programming* (R.L. Graves and P. Wolfe eds.) McGraw-Hill, New York (1963) 269-302.

[7] P.M. Gruber, C.G. Lekkerkerker, *Geometry of Numbers, second edition*, North-Holland Mathematical Library, Amsterdam (1987).

[8] M. Jörg, k-disjunctive cuts and cutting plane algorithms for general mixed integer linear programs, Ph.D. Thesis, Technische Universität München, München (2008).

[9] J.H. Owen, S. Mehrotra, A disjunctive cutting plane procedure for general mixed integer linear programs, *Mathematical Programming (Series A)* **89** (2001) 437-448.

[10] R.T. Rockafellar, R.J. Wets, *Variational Analysis*, Grundlehren der mathematischen Wissenschaften 317, Springer-Verlag, Berlin (1998).

[11] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Cambridge University Press, Cambridge (1993).

[12] A. Schrijver, On cutting planes, *Annals of Discrete Mathematics* **9** (1980) 291-296.

[13] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, Chichester (1986).